Linearization

Assume a nonlinear time-invariant (NLTI) system: $\dot{x}(t) = f(x(t))$

with an equilibrium point: $x^\star \ (f(x^\star) = 0)$

Let us linearize the vector field $f(x)$ around $x^\star$:

Note: $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ → state of n-dimensional dynamic system

$$f\big(x(t)\big) = \begin{bmatrix} f_1(x_1(t), x_2(t), \cdots, x_n(t)) \\ f_2(x_1(t), x_2(t), \cdots, x_n(t)) \\ \vdots \\ f_n(x_1(t), x_2(t), \cdots, x_n(t)) \end{bmatrix}$$ → n nonlinear functions

Short notation: $f\big(x(t)\big) = \begin{bmatrix} f_1(x(t)) \\ f_2(x(t)) \\ \vdots \\ f_n(x(t)) \end{bmatrix}$

$$\underbrace{\frac{dx(t)}{dt} = f\big(x(t)\big)}_{\text{NLTI system}} \xrightarrow{\text{linearization}} \underbrace{\frac{dz(t)}{dt} = Az(t)}_{\text{LTI system}}$$

Major questions:

Q1: By analyzing the stability of a linearized model (LTI system), what can be said about the stability of the original (NLTI) system? E.g. if the linearized model is stable, is the nonlinear model stable too?

Q2: By analyzing the type of equilibria of linearized model, what can be said about the type of equilibria of the original (NLTI) system? E.g. if the linearized equilibrium is of a certain type, say a node, is the equilibrium of the original system also a node.

$dx(t)/dt = f(x(t))$ find equilibrium $x^\star$ such that $f(x^\star) = 0$.

Linearize $f(x)$ around $x^\star$ using Taylor series expansion. This can be done for any number of dimensions n.

$$dx(t)/dt = Df(x^\star)\,[x(t) - x^\star] + \mathrm{H.O.T.}$$
$$z(t) := x(t) - x^\star$$
$$dz(t)/dt = Df(x^\star)\,z(t) = A\,z(t) \quad (\mathrm{H.O.T.\,neglected})$$

$Df(x^\star)$ – Jacobian matrix ($n\ x\ n$)

Q1: By analyzing $dz(t)/dt = A\,z(t)$, what can be said about the stability of the system $dx(t)/dt = f(x(t))$?

If $z^\star$ is a hyperbolic equilibrium (no eigenvalue of *A* has zero real part), we say that the stability of $z^\star$ determines the stability of $x^\star$, and in turn, the stability of the nonlinear system $dx(t)/dt = f(x(t))$.
If $z^\star$ is a non-hyperbolic equilibrium, then nothing can be said about stability (H.O.T. are important).

If we restrict ourselves to $n = 2$ (2-D or 2nd order systems) even stronger results are possible, but conditions are more restrictive.

Q2: By analyzing the phase portrait of $dz(t)/dt = A\,z(t)$, what can be said about the phase portrait of the system 2-D nonlinear system:
$dx(t)/dt = f(x(t))$ (at least locally around $x^\star$)?

As long as $z^\star$ is not one of the borderline cases (line, center, star, degenerate node), i.e. $z^\star$ is a saddle point, node or focus, $x^\star$ is also a saddle point, node or focus. In other words the phase portraits of $dz(t)/dt = A\,z(t)$ and $dx(t)/dt = f(x(t))$ are qualitatively similar (at least locally around $x^\star$).

If $z^\star$ is one of the borderline cases, H.O.T. are important and cannot be neglected.

Example: van der Pol Oscillator (we will demonstrate both of these points).

$$\ddot{y}(t) + \mu(y^2(t) - 1)\dot{y}(t) + y(t) = 0$$

$$\ddot{y}(t) + \mu(y^2(t) - 1)\,\dot{y}(t) + y(t) = 0$$

$$x_1(t) \coloneqq y(t)$$

$$x_2(t) \coloneqq \dot{y}(t)$$

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t)$$

$$\begin{aligned}\dot{x}_2(t) &= \ddot{y}(t) = -\mu(y^2(t) - 1)\dot{y}(t) - y(t)\\ &= -\mu(x_1^2(t) - 1)x_2(t) - x_1(t)\end{aligned}$$

$$f_1(x_1, x_2) = x_2$$

$$f_2(x_1, x_2) = -\mu(x_1^2 - 1)x_2 - x_1$$

Equilibrium point:

$$f(x^\star) = 0$$

$$f_1(x_1^\star, x_2^\star) = \boxed{x_2^\star = 0}$$

$$f_2(x_1^\star, x_2^\star) = -\mu\left({x_1^\star}^2 - 1\right)\underbrace{x_2^\star}_{0} - x_1^\star = 0 \Rightarrow \boxed{x_1^\star = 0}$$

Therefore:

$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is and equilibrium point

$\boxed{f_1(x_1, x_2) = x_2}$ already linear (nothing to linearize)

$$f_2(x_1, x_2) = -\mu(x_1^2 - 1)x_2 - x_1$$

$$f_2(x_1, x_2) = \underbrace{f_2(x_1^*, x_2^*)}_{0} + \begin{bmatrix} \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x^*} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

$$f_2(x_1, x_2) = \begin{bmatrix} -2\mu x_1 x_2 - 1 & -\mu(x_1^2 - 1) \end{bmatrix}_{x^*} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$f_2(x_1, x_2) = \begin{bmatrix} -1 & \mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{f_2(x_1, x_2) = -x_1 + \mu x_2}$$

Substitution: $z_1 = x_1;\ z_2 = x_2$

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}}_{A} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

$$\det(sI - A) = \begin{vmatrix} s & -1 \\ 1 & s-\mu \end{vmatrix} = s^2 - \mu s + 1 = 0$$

$$\boxed{s_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}}$$

<u>Case I</u> $\mu = 1 \Rightarrow s_{1,2} = \frac{1 \pm j\sqrt{3}}{2}$

$Re(s_{1,2}) \neq 0$ -hyperbolic equilibrium

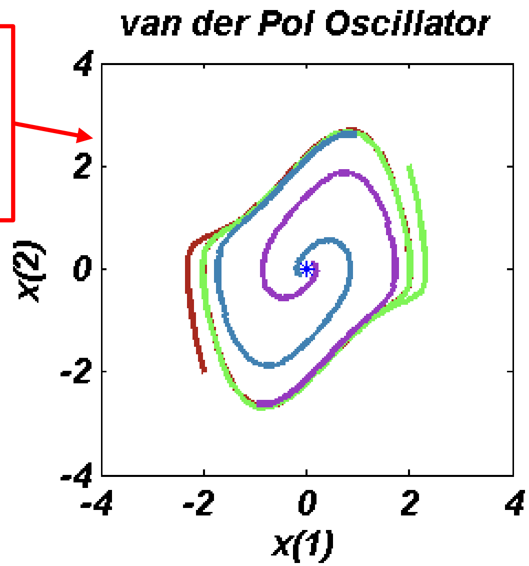
Complex conjugate poles and $Re(s_{1,2}) > 0 \Rightarrow z^*$ is an unstable focus (spiral)

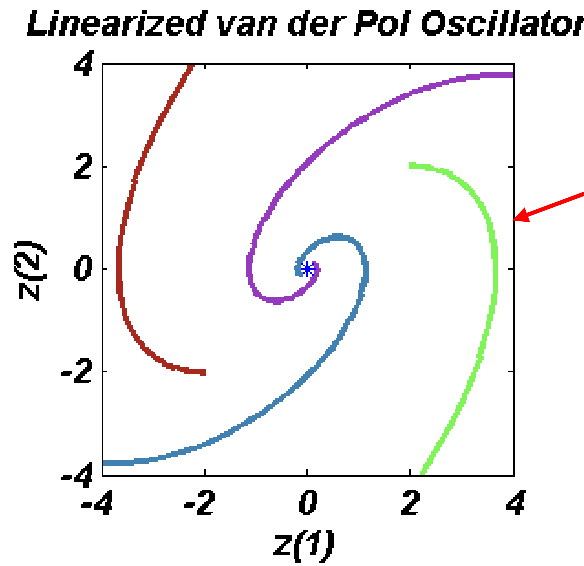
It follows that x^* is also an unstable focus (spiral), moreover, at least locally around $(0,0)$, the phase portraits of the linearized and the original system look similar!

Play with `linearized_vanderpol.m`

$\mu = 1$

unstable focus (locally)

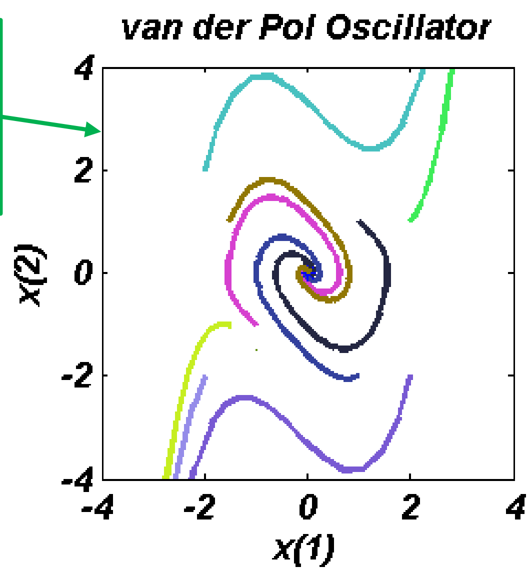


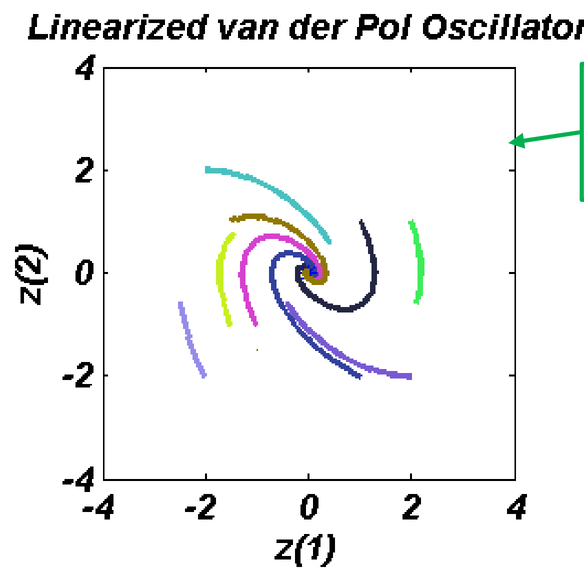


unstable focus

$\mu = -1$

stable focus (locally)





stable focus

Case II $\mu = 2; s_{1,2} = \frac{2 \pm \sqrt{4-4}}{2} = 1$

$Re(s_{1,2}) \neq 0$ -hyperbolic equilibrium

Do we have 2 linearly independent eigenvector?

We do not have 2 linearly independent eigenvectors*

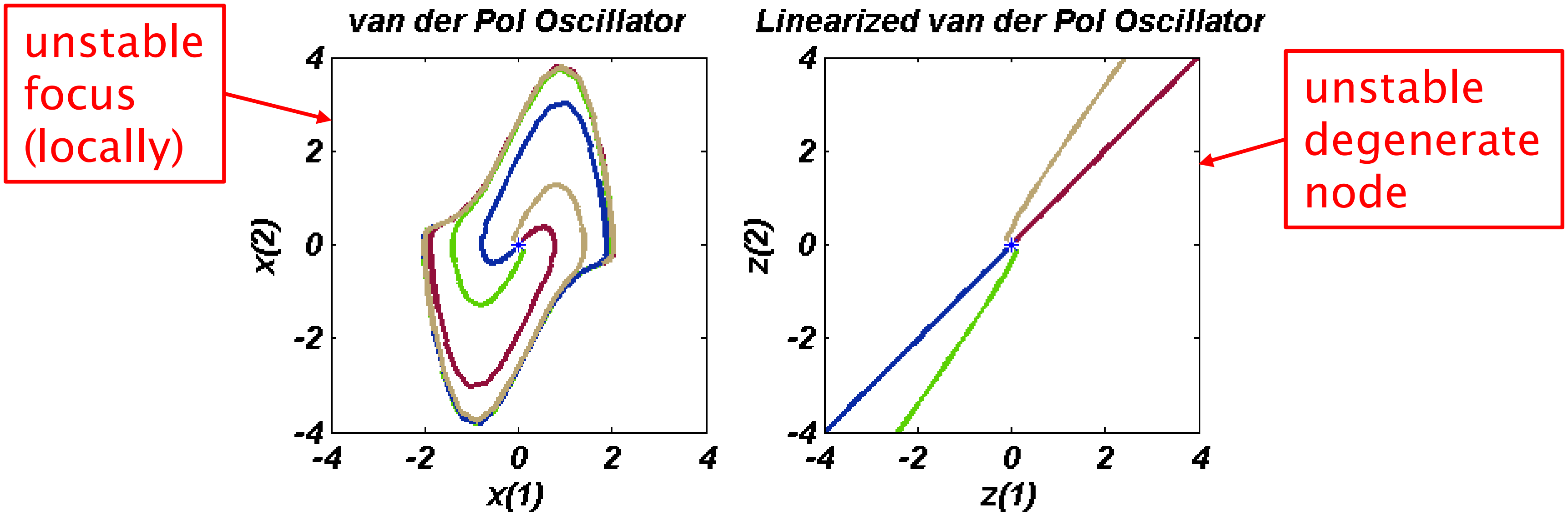
$z^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a degenerate node (unstable since $Re(s_{1,2}) > 0$)

$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is also unstable

But we have no guarantee that x^* is a degenerate node. In fact, it is not (it's an unstable focus). Locally, x^* and z^* do not look similar.

Summary: For stability based on non-hyperbolic equilibria, H.O.T. are important (linearization alone is not sufficient). In addition, H.O.T. are also important (i.e. linearization alone is not sufficient) when investigating the local phase portrait properties of borderline cases.

$\mu = 2$



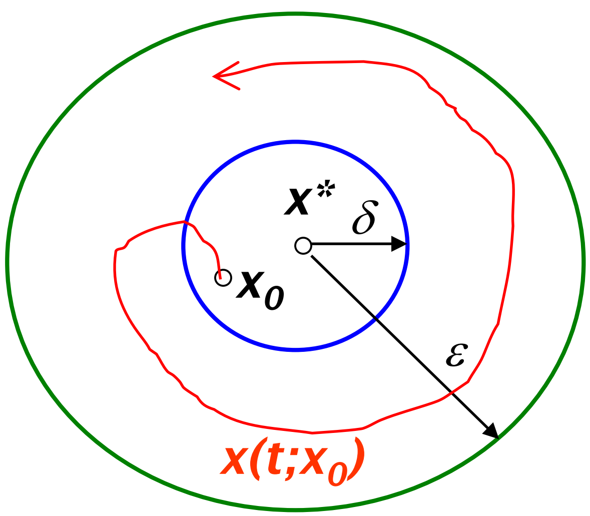
unstable focus (locally)
van der Pol Oscillator
x(2)
x(1)
4
2
0
-2
-4
Linearized van der Pol Oscillator
z(2)
z(1)
unstable degenerate node

Stability

$$\dot{x}(t) = f\big(x(t)\big) \qquad x(0) = x_0 \qquad\qquad (1)$$

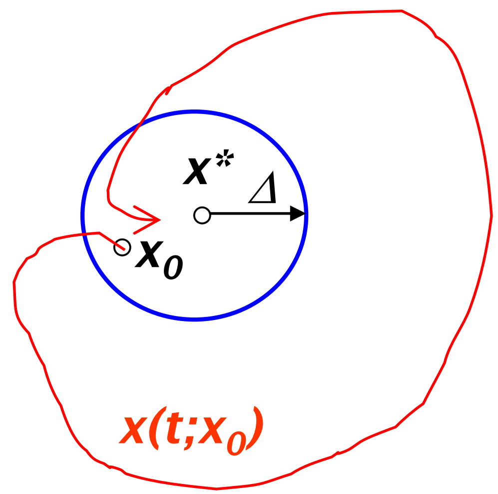
Definition [Lyapunov stability]: Let x^* be an equilibrium state of system (1) ($f(x^*) = 0$). We say that x^* is <u>Lyapunov stable</u> if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$, such that for every x_0 that satisfies $\|x_0 - x^*\| < \delta$ we have $\|x(t; x_0) - x^*\| < \varepsilon$ for all $t \geq 0$.

Note: $\|x\|$ represents the Euclidean norm.



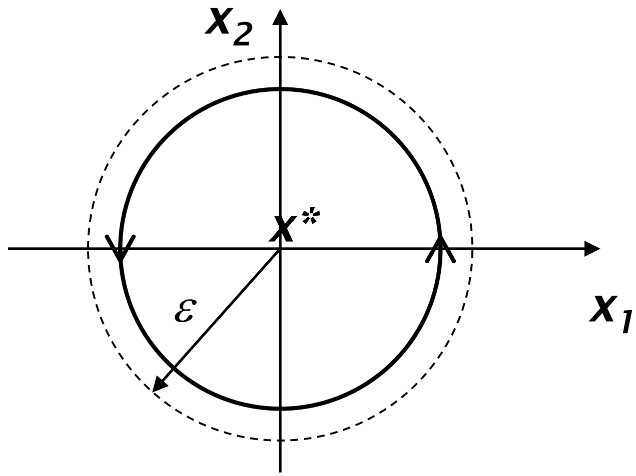
Definition: Let x^* be an equilibrium state of system (1) $(f(x^*)=0)$.
We say that x^* is attractive if there exists $\Delta > 0$, such that
$||x_0 - x^*|| < \Delta$ implies $\lim_{t \to \infty} x(t;x_0) = x^*$. If this is true for any $\Delta > 0$, we say x^* is globally attractive.

Consequently, x^* is called the attractor or global attractor.



The set $A(x^*)$ of all initial conditions x_0 such that $\lim_{t \to \infty} x(t;x_0) = x^*$ is called the region of attraction of x^* .

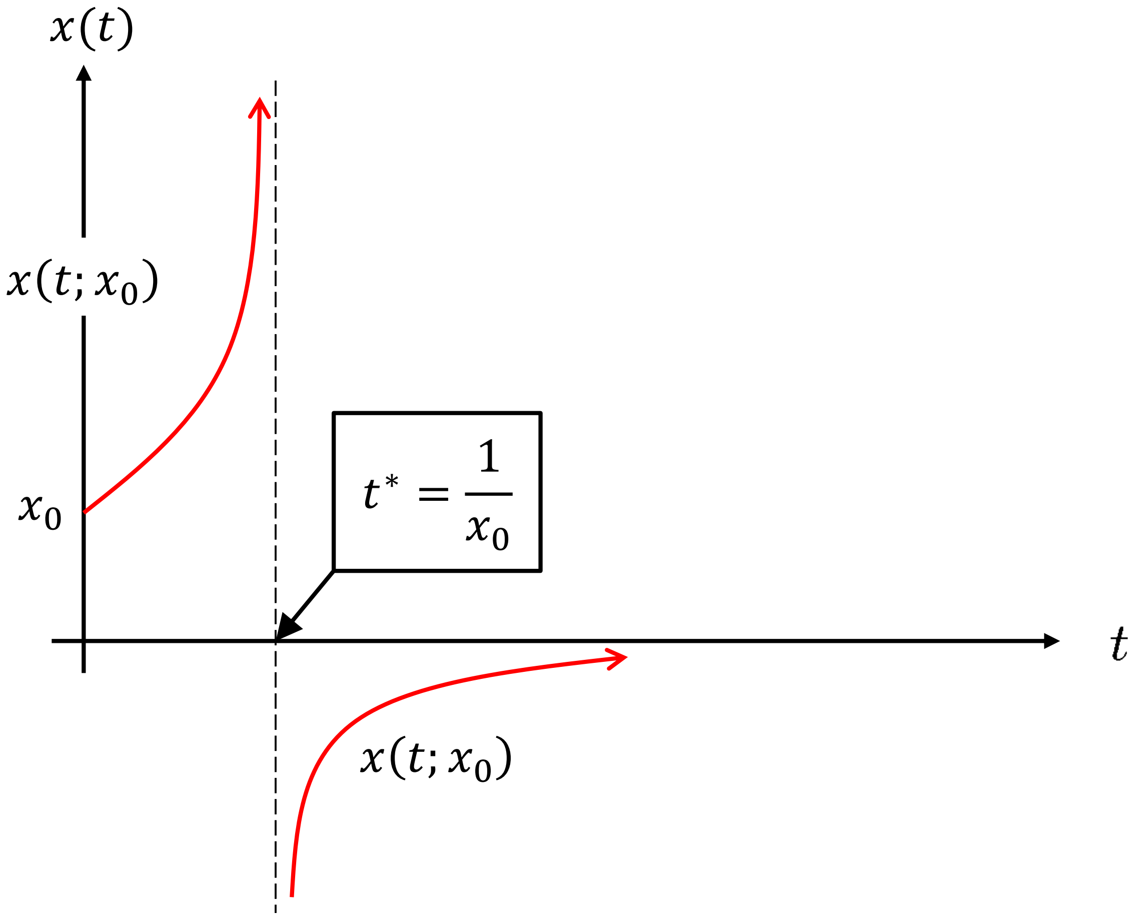
Note: If x^* is Lyapunov stable, it need not be attractive. Example: center.



$$\begin{aligned} \delta &= \varepsilon \\ \|x(t;x_0) - \underbrace{x^*}_{0}\| &< \varepsilon \\ \forall x_0 &: \quad \|x_0\| < \delta \end{aligned}$$

If an equilibrium x^* is attractive, it need not be Lyapunov stable. Examples are not so numerous, but here is one:

$$\dot{x}(t) = x^2(t) \qquad x(t;x_0) = \frac{x_0}{1 - tx_0}$$



Definition: Let x^* be an equilibrium state of system (1) $(f(x^*)=0)$. We say that x^* is asymptotically stable if:

i) x^* is Lyapunov stable
ii) x^* is attractive

Definition: Let x^* be an equilibrium state of system (1) $(f(x^*)=0)$. We say that x^* is globally asymptotically stable if:

i) x^* is Lyapunov stable
ii) x^* is globally attractive

Definition: The system (1) is stable if its equilibrium state x^* is globally asymptotically stable.