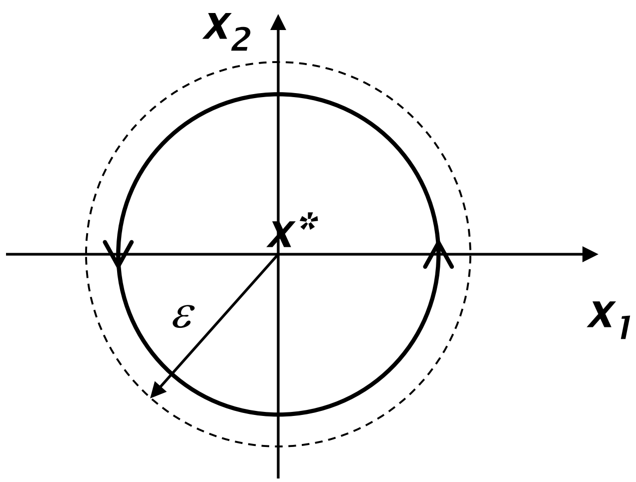
From last time:

If an equilibrium x^* is Lyapunov stable it need not be attractive. An equilibrium that is Lyapunov stable, but not attractive is called <u>marginally (neutrally) stable</u>. Examples are numerous. Any system that has center as its equilibrium.

Example: center.



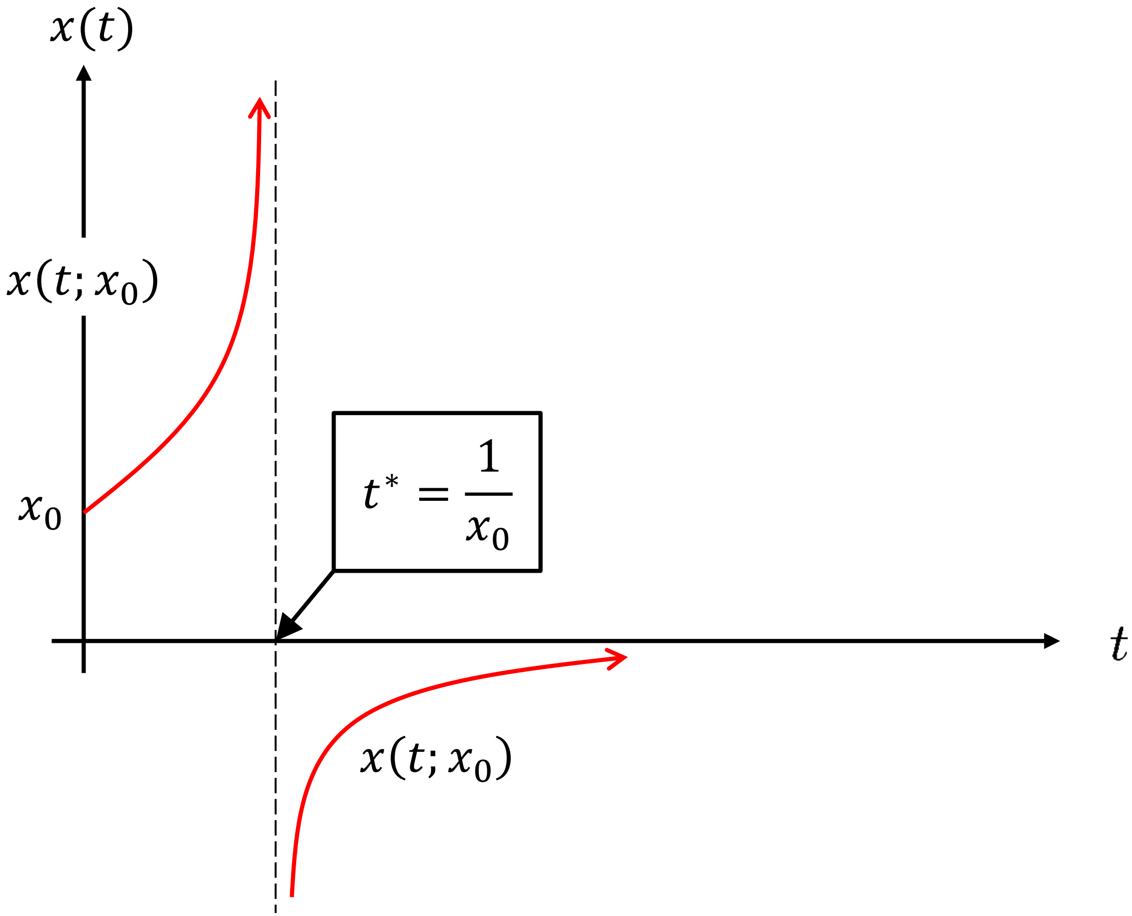
$$\delta = \varepsilon$$

$$\left\| x(t;x_0) - \underbrace{x^*}_{0} \right\| < \varepsilon$$

$$\forall x_0 : \left\| x_0 \right\| < \delta$$

If an equilibrium x^* is attractive, it need not be Lyapunov stable. Examples are not so numerous, but here is one:

$$\dot{x}(t) = x^2(t) \qquad\qquad x(t; x_0) = \frac{x_0}{1 - t x_0}$$



Isoclines

Recall: $$\underbrace{dx(t)/dt}_{\text{velocity}} = f(x(t))$$

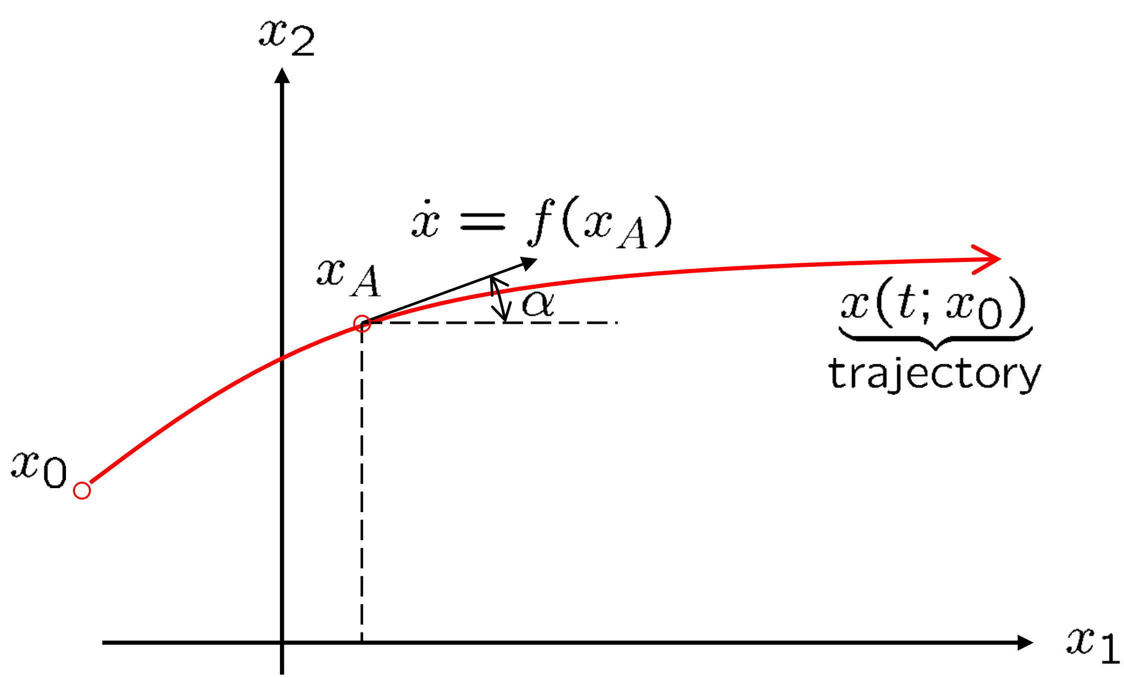
Also note that $f\big(x(t)\big) \in \mathbb{R}^n$, therefore $f(x(t))$ is a vector field.

Since $f(x(t))$ is parameterized by t, it is often called the flow of the vector field.

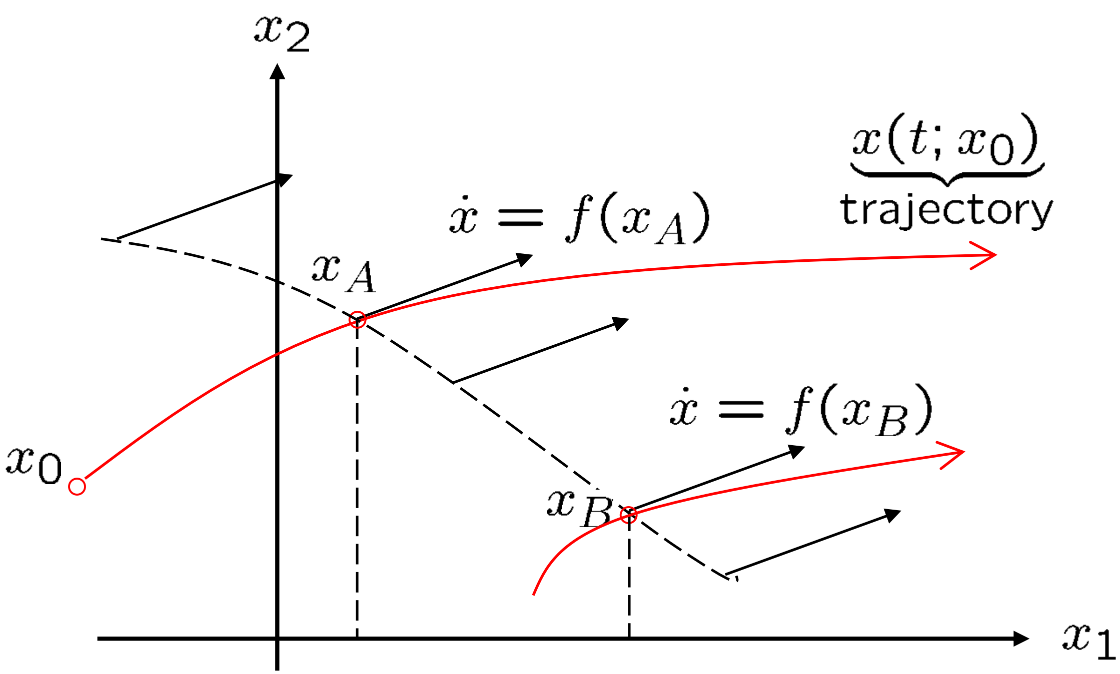
Special case: $n = 2$

$$\frac{dx(t)}{dt} = f\big(x(t)\big)$$
$$x(t) \in \mathbb{R}^2$$
$$x(0) = x_0$$

$\tan\alpha$ = slope of the vector field at $x = x_A$



What if I find another point x_B such that: $slope\ @\ x_B = slope\ @\ x_A$?



Then x_B and x_A belong to a curve called <u>isocline</u>.

A collection of states x such that $Slope\big(f(x)\big) = Const. = C$ is called an isocline.

Note: isocline is not necessarily a trajectory.

What are they useful for?

Isoclines are useful for sketching phase portraits of 2-D systems.

They are also easy to calculate.

Slope of $f(x)$ is determined by:

$$
\begin{aligned}
\text{Slope} &= \text{Const.} = C \\
\text{Slope} &= \frac{f_2(x)}{f_1(x)} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{dx_2}{dx_1} \\
\frac{dx_2}{dx_1} &= \underbrace{C = \frac{f_2(x)}{f_1(x)}}_{\text{eq. of isocline}}
\end{aligned}
$$

Example: $$\frac{d^2y(t)}{dt^2} - 0.5\,y(t)\frac{dy(t)}{dt} + y(t) = 0$$

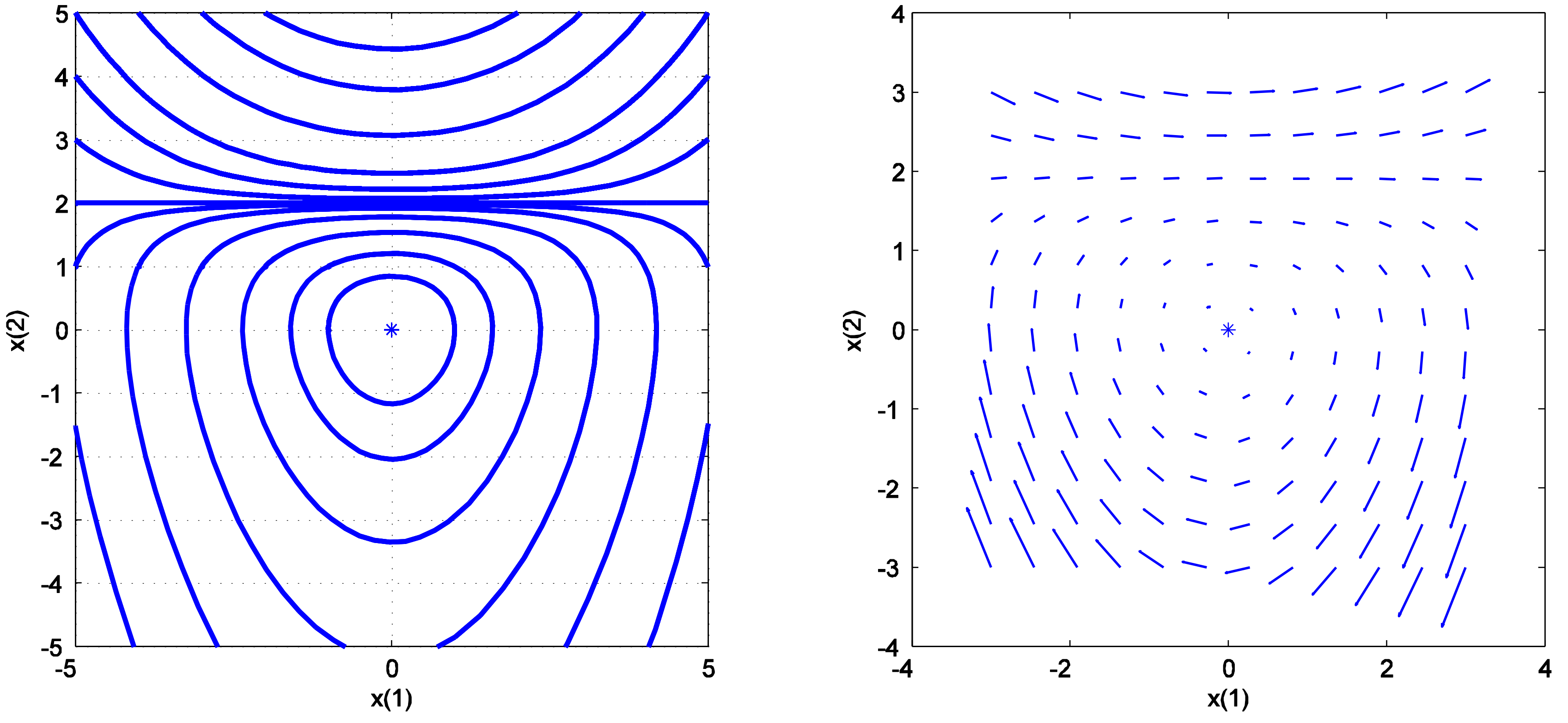
$$x_1(t) \coloneqq y(t)$$
$$x_2(t) \coloneqq \frac{dy(t)}{dt}$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = 0.5x_1(t)x_2(t) - x_1(t)$$

Isocline: $\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \boxed{\frac{0.5x_1x_2 - x_1}{x_2} = C}$ implicit equation

Similarly: $x_2(C - 0.5x_1) = -x_1 \Rightarrow \boxed{x_2} = -\frac{x_1}{C - 0.5x_1} = \frac{x_1}{0.5x_1 - C} = \boxed{\frac{2x_1}{x_1 - 2C}}$

explicit equation



5
4
3
2
1
0
-1
-2
-3
-4
-5
x(2)
-5
0
5
x(1)
4
3
2
1
0
-1
-2
-3
-4
x(2)
-4
-2
0
2
4
x(1)

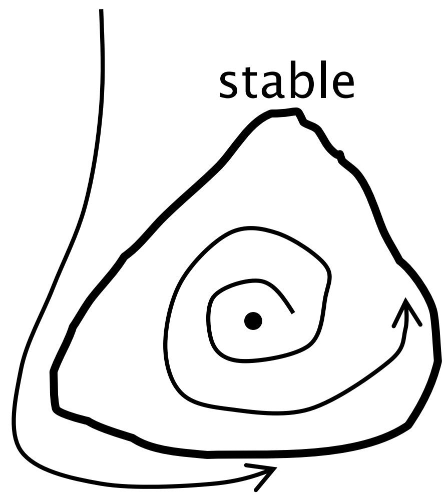
Limit Cycle

Limit cycle (periodic orbit) is an isolated closed trajectory in the phase space.

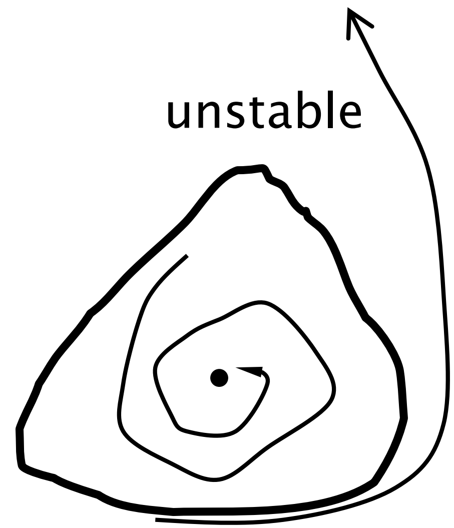
Limit cycle can be:

1. stable
2. unstable
3. half-stable

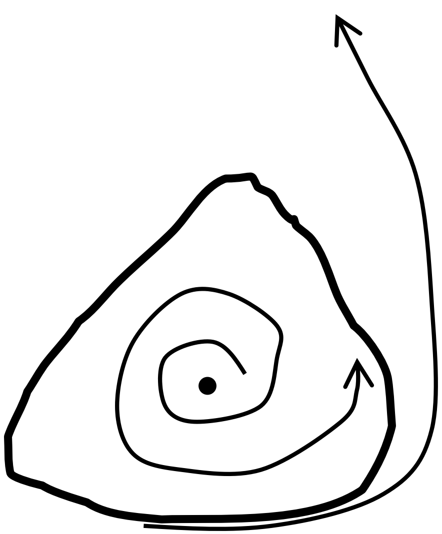
stable

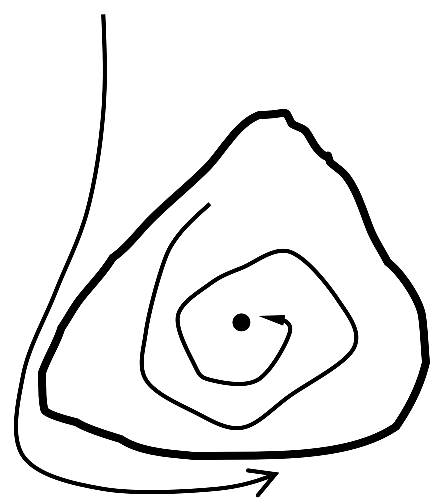


unstable



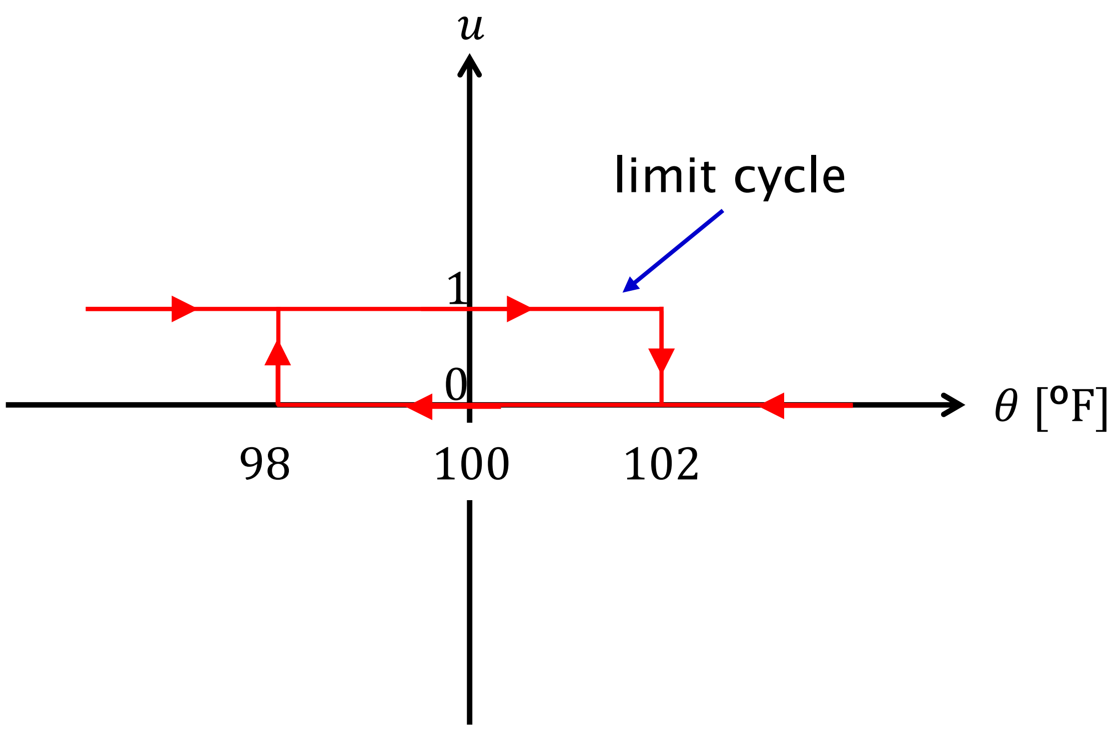
half-stable



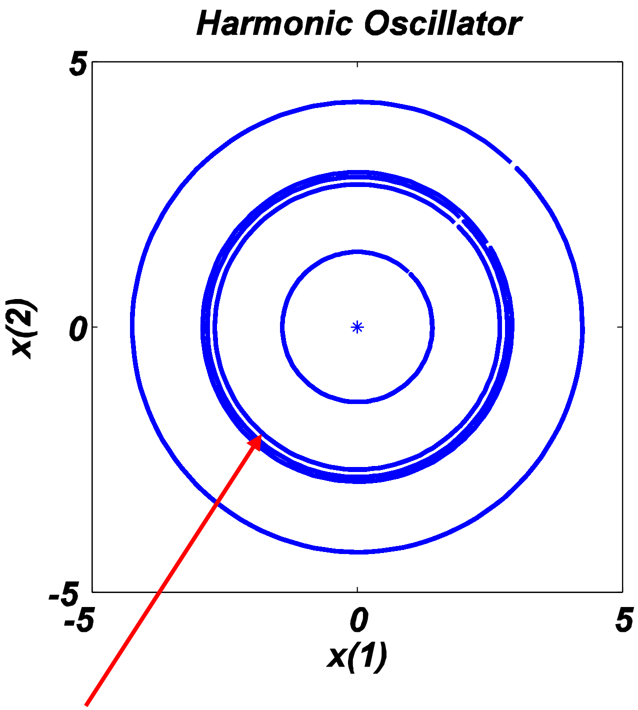


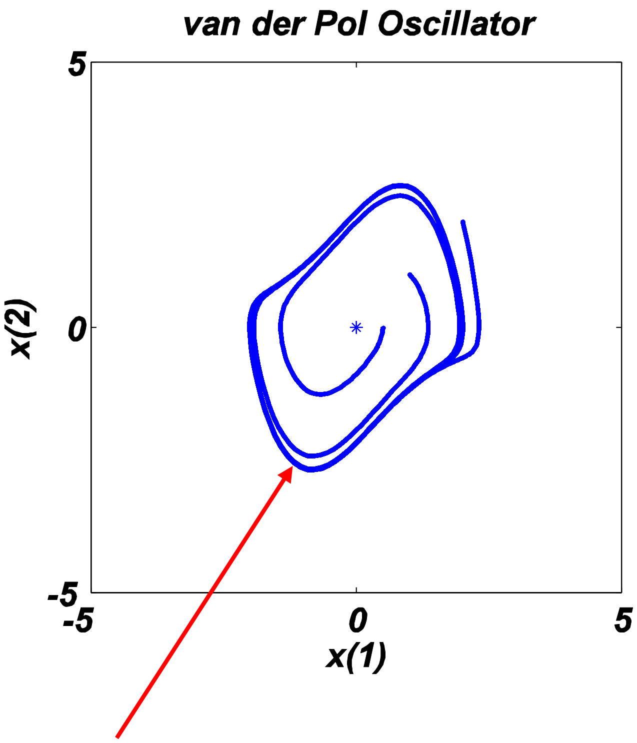
Limit cycle is the hallmark of nonlinear oscillations.

Here is a simple example. Water is heated to 100ºF, and a control system is to maintain this temperature. However the heating system is not perfectly sensitive, i.e. it turns on ($u = 1$) only when temperature drops to 98ºF, and it shuts off ($u = 0$) when temperature reaches 102ºF. Assuming the outside temperature is <98ºF, this system will evolve according to the limit cycle.



Note: Not every closed orbit is a limit cycle.





closed but not isolated
(not a limit cycle)

closed and isolated (limit cycle)

Homoclinic orbit: $\ddot{y}(t) = y(t) - y^3(t)$

State variables: $x_1 = y; x_2 = \dot{y}$

State-space form:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3$$

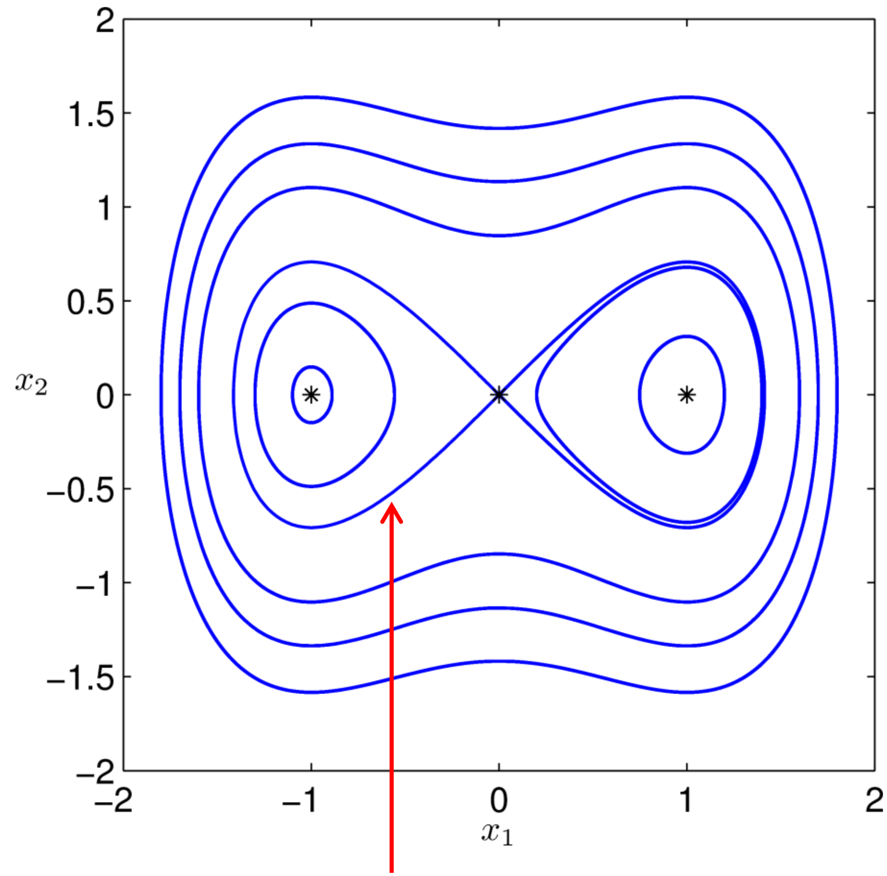
Equilibria:

$$x_1^* = \begin{bmatrix}0\\0\end{bmatrix}, x_2^* = \begin{bmatrix}1\\0\end{bmatrix}, x_3^* = \begin{bmatrix}-1\\0\end{bmatrix}$$

Linearization: $A = \begin{bmatrix}0 & 1\\ 1-3x_1^2 & 0\end{bmatrix}$

$$A_1 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \lambda_{1,2} = \pm 1, v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$$

`homoclinic_orbit.m`

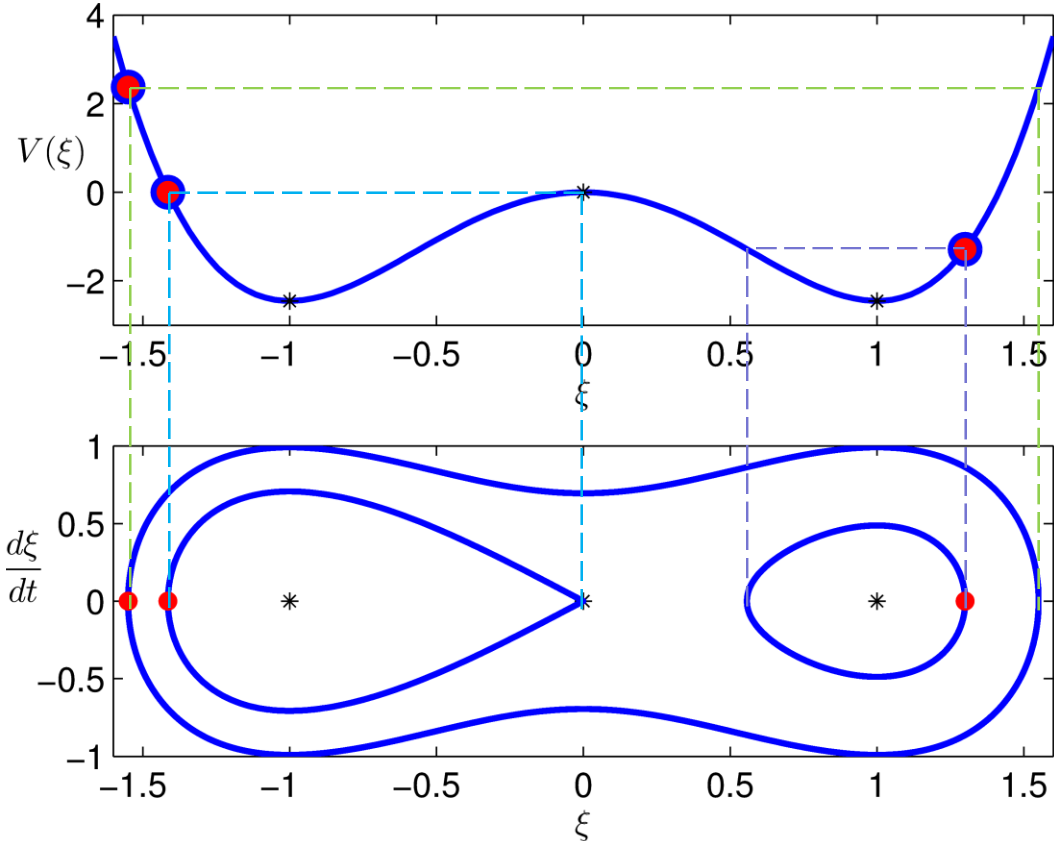


homoclinic orbit:
$$\lim_{t\to\infty} x(t;x_0) = \lim_{t\to-\infty} x(t;x_0)$$

Movement of a ball in a double (potential) energy well:

Potential:

$$V(\xi) = -\frac{g}{2}\xi^2 + \frac{g}{4}\xi^4$$



$$E_k + E_p = const. \quad (\text{no frict.})$$

$$\frac{m}{2}\dot{\xi}^2 + mV(\xi) = const.$$

$$2\frac{m}{2}\dot{\xi}\ddot{\xi} + m\frac{\partial V(\xi)}{\partial \xi}\dot{\xi} = 0$$

$$\dot{\xi}\ddot{\xi} + [-g\xi + g\xi^3]\dot{\xi} = 0$$

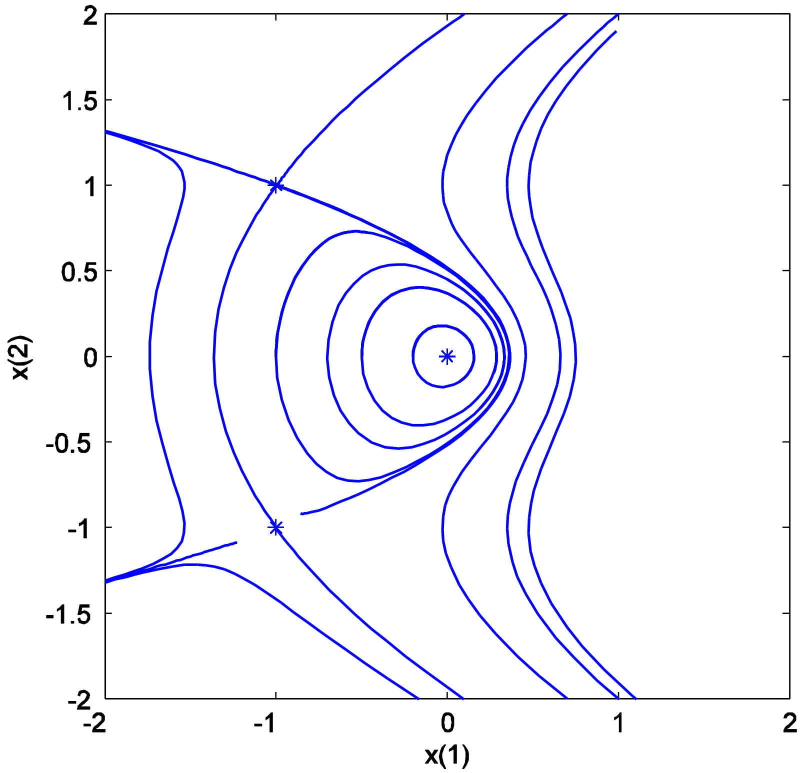
Either: $\dot{\xi} = 0, (\text{equilibrium})$

or: $\ddot{\xi} = g\xi - g\xi^3$ (very similar to the original system, the same if $g = 1$)

If you feel adventurous, show that: $x_2 \approx \pm\sqrt{x_1^2 - \frac{x_1^4}{2}}$ is the eq. of the homoclinic orbit.

Heteroclinic orbit:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) - x_2^3(t)\\ \dot{x}_2(t) &= -x_1(t) - x_2^2(t)\end{aligned}$$



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