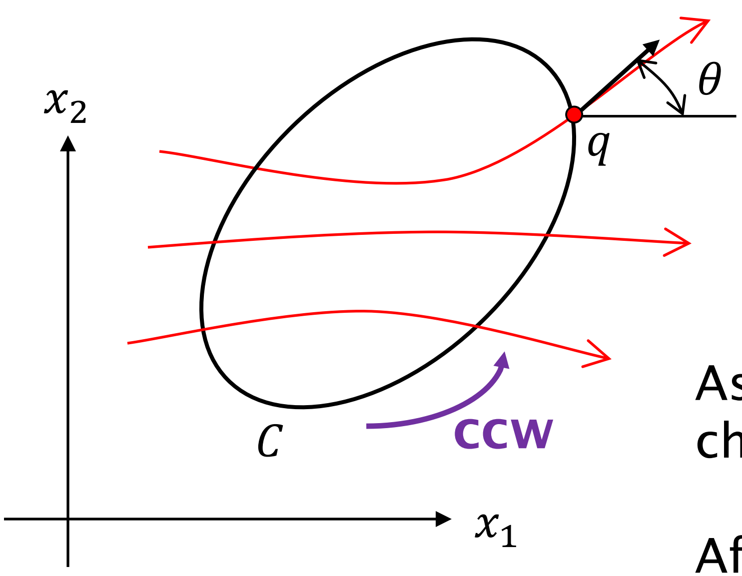
Elementary Index Theory

Second order ($n = 2$) nonlinear time-invariant dynamic system:

$$\frac{dx(t)}{dt} = f\big(x(t)\big)$$

$f = [f_1, \ f_2]$ – 2-D vector field. Assume f is smooth.

C – a simple closed curve in $\mathbb{R}^2$ that does not pass through any equilibrium points. C is not necessarily a trajectory.



At each point q on the curve define: $\theta = \theta(q)$ –the angle of the slope of the vector field at point q:

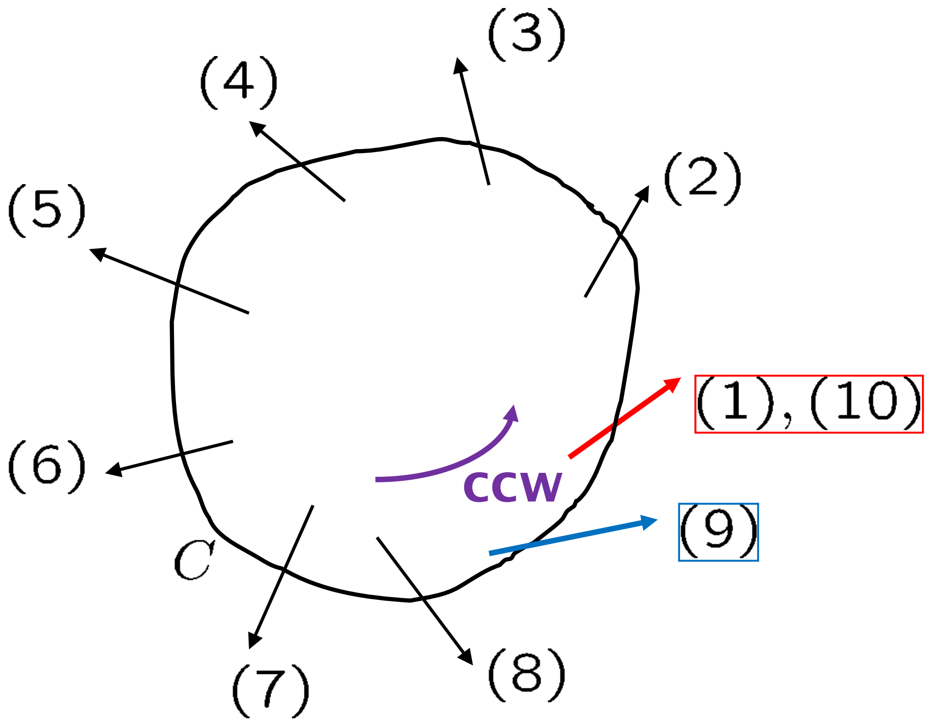
$$\theta(q) \coloneqq \tan^{-1} \left.\frac{\dot{x}_2}{\dot{x}_1}\right|_{x=q}$$

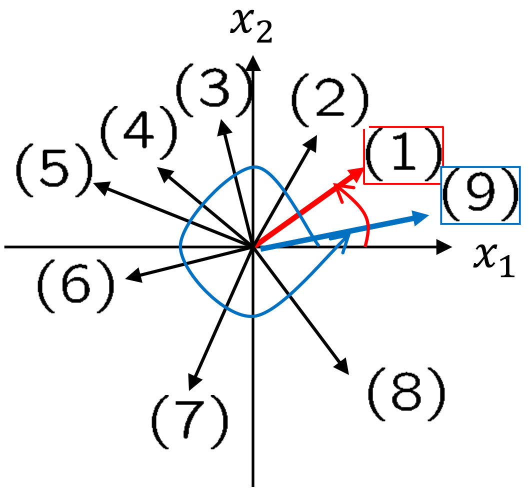
As q moves CCW around C, the angle θ changes continuously (why?)

After one revolution, q returns to the initial point q_0 where $\theta = \theta(q_0)$.

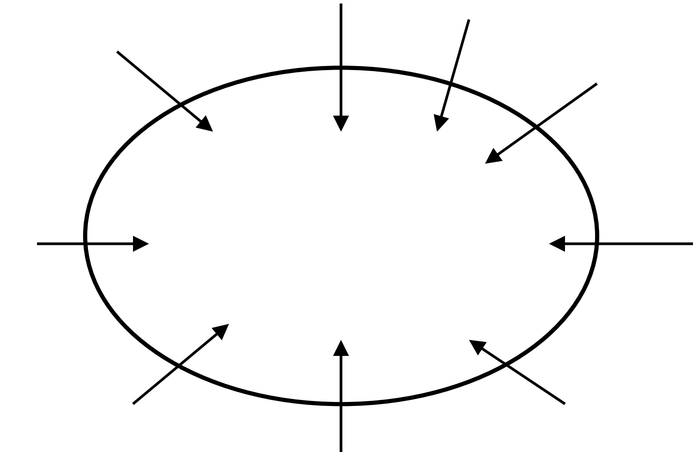
If $[\theta]$ is the net change of θ over one CCW revolution, the index of C is defined as:

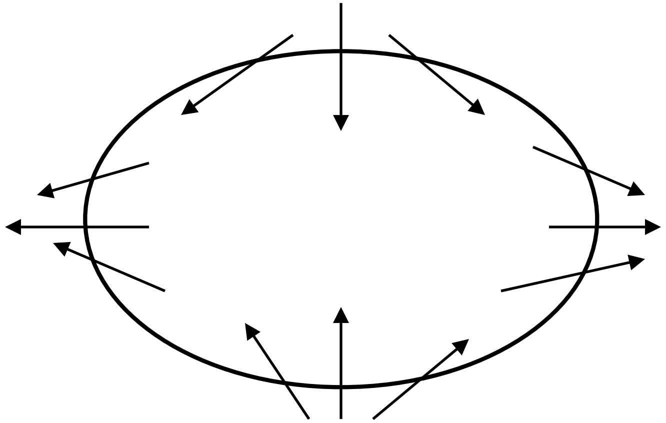
$$\boxed{I_C := \frac{1}{2\pi}[\theta]}$$

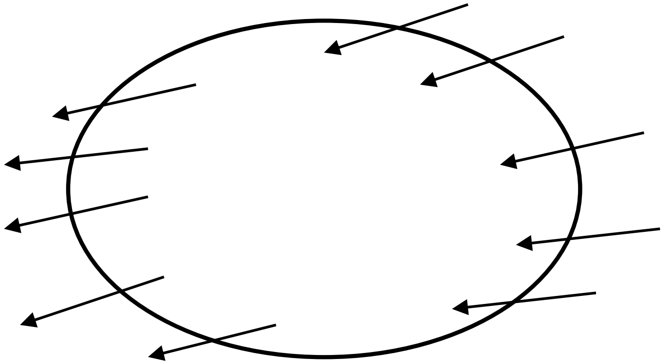




$$I_C = \frac{1}{2\pi}2\pi = 1$$







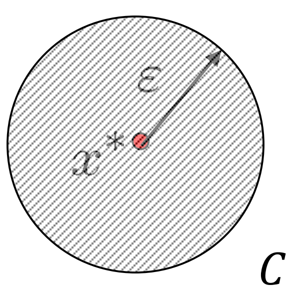
Properties of the Index

1) If C does not encircle an equilibrium point, then $I_C = 0$.

2) Let $C1$ and $C2$ be simple closed curves which do not pass through any equilibria. Suppose that $C2$ lies entirely in $C1$ and that there are no equilibria in between. Then $I_{C1} = I_{C2}$.

Index of a point

Suppose x^* is an <u>isolated</u> equilibrium point. The index of x^* is the index of a circle $C := \{x: \|x - x^*\| = \varepsilon\}$, where ε is chosen so small that no other equilibria lie in $\|x - x^*\| \leq \varepsilon$.



$$\|x - x^*\| = \varepsilon \Leftrightarrow \underbrace{\sqrt{(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2} = \varepsilon}_{\text{circle with radius } \varepsilon}$$

Theorem: The index of a closed curve C is equal to the sum of the indices of isolated equilibria inside of C.

Theorem: The index of a focus, node, star, center and degenerate node is +1.

Theorem: The index of a saddle point is -1.

Theorem: The index of any closed trajectory (and a periodic orbit) is +1.

Example: Show that the system

$$\frac{dx_1}{dt} = x_1(4 - x_2 - x_1^2)$$
$$\frac{dx_2}{dt} = x_2(x_1 - 1)$$

has no periodic orbit.

Solution:

Equilibrium points:

$$x_1(4 - x_2 - x_1^2) = 0$$
$$x_2(x_1 - 1) = 0$$

$(x_1 = 0 \text{ or } 4 - x_2 - x_1^2 = 0) \text{ and } (x_2 = 0 \text{ or } x_1 = 1)$

$$x^\star = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x^\star = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, x^\star = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, x^\star = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$Df(x) = \begin{bmatrix} 4 - x_2 - x_1^2 - 2x_1^2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix} = \begin{bmatrix} 4 - x_2 - 3x_1^2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}$$

$Df\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = 4$ (hyperbolic eq., saddle point $I = -1$)

Eigenvectors: $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

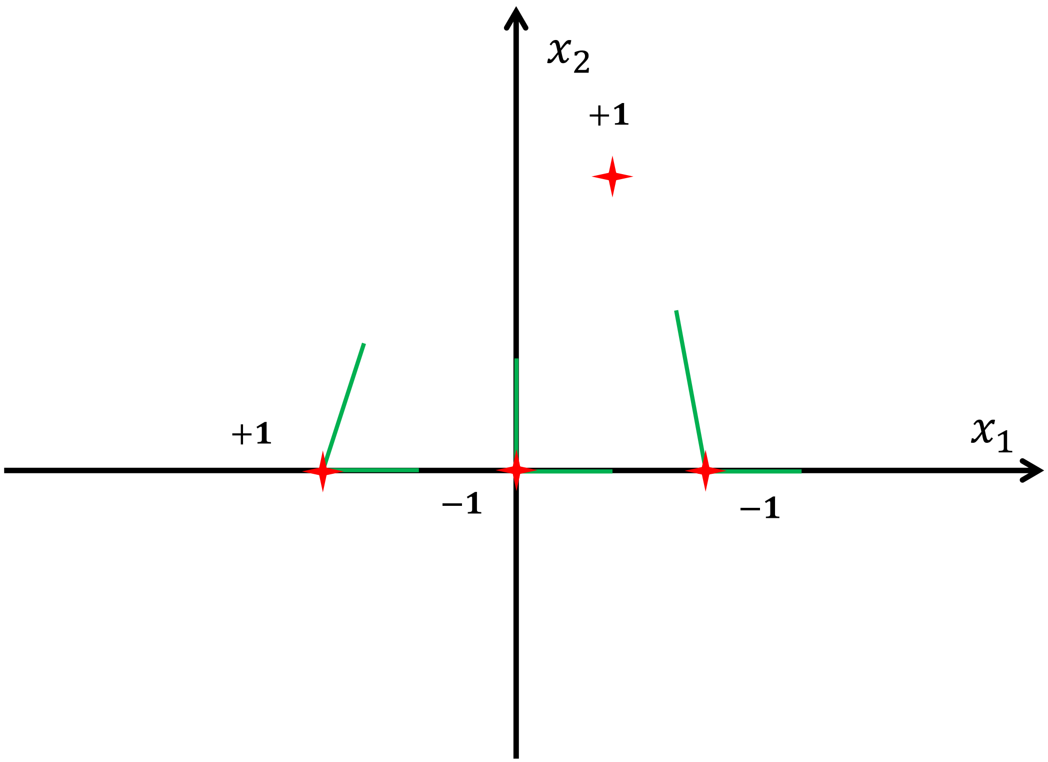
$Df\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -8 & -2 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = -8, \lambda_2 = 1$ (hyperbolic eq., saddle point $I = -1$)

Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -4.5 \end{bmatrix}$

$Df\left(\begin{bmatrix}-2\\0\end{bmatrix}\right) = \begin{bmatrix}-8 & 2\\0 & -3\end{bmatrix} \Rightarrow \lambda_1 = -8, \lambda_2 = -3$ (hyperbolic eq., stable node $I = +1$)

Eigenvectors: $v_1 = \begin{bmatrix}1\\0\end{bmatrix}, v_2 = \begin{bmatrix}1\\2.5\end{bmatrix}$

$Df\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-2 & -1\\3 & 0\end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm j\sqrt{2}$, (hyperbolic eq., stable focus $I = +1$)



Check calculations with `rule_out_closed_orbit.m`

Check phase portrait with `no_orbit_pp.m`