Intro Linear Algebra 3A: midterm 1

Monday January 29 2018, 3:00 – 3.50 pm

There are 5 exercises, worth a total of 40 points.
No calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (12 = 2 + 5 + 3 + 2 pts)
Let

$$A = \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}.$$

(a) Compute $A\mathbf{u}$.
(b) Compute the reduced row echelon form of the augmented matrix $[A|\mathbf{b}]$.
(c) Solve $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
(d) Solve $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

Solution:
(a) $[4, 4, -2]^T$.
(b)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right].$$

(c) $[1, 2, 0, 0]^T + x_4[0, -1, 1, 1]^T$.
(d) $x_4[0, -1, 1, 1]^T$.

Exercise 2 (8 = 2 + 2 + 2 + 2 pts)
Let $c \in \mathbf{R}$. Consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ c \end{bmatrix}.$$

(a1) For which c are the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ linearly dependent?
(a2) For each c as in part (a1) find a dependence relation.
(b1) For which c is the span of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ not equal to $\mathbf{R}^4$?
(b2) For each c as in part (b1) find a vector which is not in the span of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

Solution:
(a1) $c = 1$.
(a2) $\mathbf{u}_3 - \mathbf{u}_4 = 0$.
(b1) $c = 1$.
(b2) Any vector $[x_1, x_2, x_3, x_4]^T$ with $x_3 \neq x_4$ will do, such as $[0, 0, 0, 1]^T$.

Exercise 3 (6 = 2 + 2 + 2 pts)
Consider the matrix and vector

$$A = \begin{bmatrix} 2 & 0 & 3 & 4 \\ 0 & 5 & 0 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear map which sends $\mathbf{x}$ to $A\mathbf{x}$.
(a) Compute $T(\mathbf{u})$.
(b) Is T onto?
(c) Is T one-to-one?

Solution:
(a) $[9, 6, 7]^T$.
(b) Yes, pivot in every row.

(c) No, free variables.

Exercise 4 (4 pts)
Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear map such that

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \; T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

Compute the standard matrix of T.

Solution:

$$\begin{bmatrix} 0 & 1 & 1 \\ -4 & 2 & 4 \\ 3 & 3 & 0 \end{bmatrix}.$$

Exercise 5 (10 pts)
True or false? **No** explanation required. Each question is worth 1 point.
(1) There is a 10×8 matrix with 9 pivots.
(2) Every matrix has a unique reduced row echelon form.
(3) Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^7$. Then $2(\mathbf{u} - 7\mathbf{v}) + 3\mathbf{v} = 2\mathbf{u} - 10\mathbf{v}$.
(4) The vectors $[1, 2, 3, 4, 5]^T$, $[2, 4, 6, 8, 10]^T$ and $[1, 5, 12, 1232, 1]^T$ are linearly dependent.
(5) Let A be an $n \times n$ matrix. If the equation $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution, then the span of the columns of A is not $\mathbf{R}^n$.
(6) The matrix equation $A\mathbf{x} = \mathbf{b}$ always has a solution.
(7) The matrix of reflection through the line $x_2 = -x_1$ is given by $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.
(8) Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{x}$ be vectors in $\mathbf{R}^7$. If $\mathbf{x} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{x} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
(9) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a map. Then there is a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^n$.
(10) The matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

have the same reduced row echelon form.

Solution:
(1) False.
(2) True.
(3) False.
(4) True.
(5) True.
(6) False.
(7) True.
(8) True.
(9) False.
(10) False.