Intro Linear Algebra 3A: midterm 2

Friday May 13, 2:00- 2:50pm

There are 4 exercises, worth a total of $100 = 18 + 24 + 28 + 30$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (18 pts)
Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 2 \end{bmatrix}.$$

Compute A^{-1}.

Solution:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Exercise 2 $(24 = 8 + 8 + 2 + 6$ pts)
Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 & 4 \\ 3 & 1 & 1 & 2 & 7 \\ 4 & 0 & 1 & 1 & 6 \end{bmatrix}$$

with reduced row echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute a basis of the null space $\mathrm{Nul}(A)$ of A.
(b) Compute a basis of the column space $\mathrm{Col}(A)$ of A.
(c) What are the dimensions of the null space and column space of A (2 numbers)?
(d) Do the 2nd, 3rd and 5th column form a basis of the column space of A?

Solution:
(a) Use parametric vector form: $\{[0,-1,-1,1,0]^T, [-1,-2,-2,0,1]^T\}$.
(b) Pivot columns: $\{[1,2,3,4]^T, [0,0,1,0]^T, [1,1,1,1]^T\}$.
(c) Null space is 2 dimensional, column space is 3 dimensional (see sizes of bases at a and b).
(d) Yes. If you look at B, you see that the columns span all the other columns. The same is true for the original matrix.

Exercise 3 (28 = 8 + 10 + 6 + 4 pts)
Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A (hint: 2 and -2 are the only eigenvalues of A).
(b) Compute a basis for the eigenspaces of each eigenvalue of A.
(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
(d) Is the answer in (c) unique? If no, find another P and D.

Solution:
(a) $(2-\lambda)^3(-2-\lambda)$.
(b) E_2 has basis $\{[1,0,0,0]^T, [0,1,0,0]^T, [0,0,-1,1]^T\}$ and E_{-2} has basis $\{[0,0,1,1]^T\}$.
(c) Yes, sum of dimensions of eigenspaces is 4. One can take:

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(d) There are multiple answers, one can switch the first two columns of P for example.

Exercise 4 (30 pts)
True or false? **No** explanation required. Points is $-10 + 4 \cdot \#\text{correct}$.

(1) Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Then A is diagonalizable.

(2) Let A, B be 3×3 matrices such that $\det(A) = 2$, $\det(B) = -2$. Then one has $\det(2AB^{-1}A^T) = -16$.

(3) Let A, B be $n \times n$ matrices. Assume that $3A^T B^{12}$ is invertible. Then A is invertible.

(4) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a set of linearly independent vectors in $\mathbf{R}^n$. Then there is a subspace of $\mathbf{R}^n$ for which this set forms a basis.

(5) Any plane in $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$.

(6) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation determined by a 3×3 matrix A. Let S be a region in $\mathbf{R}^3$ with finite volume v. Then $T(S)$ has volume $\det(A) \cdot v$.

(7) Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix such that $AB = I_m$. Then A is invertible.

(8) Let A be an $n \times n$ matrix with characteristic polynomial f. Then one has $f(0) = \det(A)$.

(9) Let A be an $n \times n$ matrix. Let $\mathbf{v}_1, \mathbf{v}_2$ be eigenvectors of A. Then $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector of A.

(10) Let A, B be $n \times n$ matrices with characteristic polynomials f and g. Then fg is the characteristic polynomial of AB.

Solution:
(1) False, 2 is only eigenvalue, and E_2 is only 1-dimensional.
(2) True (do the computation).
(3) True, can be seen by taking the determinant for example.
(4) True, namely for the span of $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$.
(5) False, the planes not going through 0 are not subspaces.
(6) False, it has volume $|\det(A)|v$.
(7) False, one can easily find counter examples (it is true if $n = m$), such as $[0,1][0,1]^T = [1]$.
(8) True, follows from definition of characteristic polynomial.
(9) False, many counter examples exist (in fact, if eigenvalues are different, it is never eigenvector).
(10) False, even the degree of the polynomials is incorrect if $n > 1$.