Intro Linear Algebra 3A: midterm 2

Monday February 22 2016, 3:00- 3:50pm

There are 3 exercises, worth a total of 100 points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations.

Name:

Student ID:

Exercise 1 (26 = 20 + 6 pts)
Consider the 3×3 matrix $\mathbf{A}$ and vector $\mathbf{b} \in \mathbf{R}^3$ given by

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 2 \\ 4 & 2 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}.$$

(a) Compute $\mathbf{A}^{-1}$ (warning: there will be fractions).
(b) Solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ using your answer to a.

Solution:
(a)

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & -\frac{1}{5} & \frac{2}{5} \\ -\frac{1}{2} & \frac{1}{5} & \frac{1}{10} \\ 1 & \frac{2}{5} & -\frac{4}{5} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 0 & -2 & 4 \\ -5 & 2 & 1 \\ 10 & 4 & -8 \end{bmatrix}.$$

(b) $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = [3, 1, -4]^T$.

Exercise 2 ($38 = 8 + 4 + 8 + 6 + 4 + 8$ pts)
Let $c \in \mathbf{R}$. Consider the 3×3 matrix and vector $\mathbf{b}$ given by

$$\mathbf{A}_c = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Compute the determinant of $\mathbf{A}_c$.
(b) Explain why $\mathbf{A}_c$ is invertible if and only if $c \neq 1, -2$.
(c) For $c = 1$, compute a basis for the null space and the column space of $\mathbf{A}_c$.
(d) For $c = -2$, what is $\operatorname{rank} \mathbf{A}_c$ and $\dim \operatorname{Nul} \mathbf{A}_c$?
(e) For $c \neq 1, -2$, what is $\operatorname{rank} \mathbf{A}_c$ and $\dim \operatorname{Nul} \mathbf{A}_c$?
(f) For $c = 0$, consider the equation $\mathbf{A}_c\mathbf{x} = \mathbf{b}$. Let $\mathbf{x} = [x_1, x_2, x_3]^T$ be the unique solution. Use Cramer's rule to find x_2.

Solution:
(a) $c^3 - 3c + 2 = (c-1)^2(c+2)$.
(b) See factorization in a, the determinant is nonzero.
(c) Basis for null space is $\{[-1, 0, 1]^T, [-1, 1, 0]^T\}$, and a basis for the column space is $\{[1, 1, 1]^T\}$.
(d) Rank is 2 (it is not 3, and first and second column are linearly independent), dimension null space is 1 (because dimensions of null space plus rank is 3).
(e) The matrix is invertible, rank 3, and dimension null space is 0.
(f) One has $\det(A_0) = 2$. One computes

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 2.$$

One finds $x_2 = 2/2 = 1$. In fact, the solution is $[2, 1, 0]^T$.

Exercise 3 (36 pts)
True or false? **No** explanation required. Points: correct answer 3, incorrect answer -1, no answer 0.

(1) Let $\mathbf{A}$ be an $n \times n$ matrix such that $\mathbf{A}^3 + 2\mathbf{A}^2 + 3\mathbf{A} + 4I_n = 0$. Then $\mathbf{A}$ is invertible and one has $\mathbf{A}^{-1} = -\frac{\mathbf{A}^2+2\mathbf{A}+3I_n}{4}$.

(2) Let $\mathbf{A}$ be a 3×2 matrix. Then there is never a 2×3 matrix $\mathbf{B}$ such that $\mathbf{BA} = I_2$.

(3) Let $\mathbf{A}$ be a 2×3 matrix. Then there is never a 3×2 matrix $\mathbf{B}$ such that $\mathbf{BA} = I_3$.

(4) The plane in $\mathbf{R}^3$ given by the equation $3x + 4y + 5z = 6$ is a subspace of $\mathbf{R}^3$.

(5) Consider the ordered basis $\mathfrak{B} = \{[0,1,0]^T, [0,0,1]^T, [1,0,0]^T\}$ of $\mathbf{R}^3$. Let $x = [1,2,3]^T$. Then one has $[x]_{\mathfrak{B}} = [1,2,3]^T$.

(6) Let V be a subspace of $\mathbf{R}^n$. Then one has $\dim(V) \leq n$.

(7) Let $\mathbf{A}$ be an invertible matrix with all integer entries. Then all entries of $\det(\mathbf{A})\mathbf{A}^{-1}$ are integers.

(8) Let S be a region of $\mathbf{R}^3$ with finite volume. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear map determined by a matrix $\mathbf{A}$. Then the volume of $T(S)$ is the volume of S multiplied by $|\det(\mathbf{A})|$.

(9) Let $\mathbf{A}, \mathbf{B}$ be 2×2 matrices with $\det(\mathbf{A}) = 2$, $\det(\mathbf{B}) = 3$. Then one has $\det(2\mathbf{A}^2\mathbf{B}\mathbf{A}^{-1}) = 12$.

(10) Let $\mathbf{A}$ be an $n \times n$ matrix with 2 identical columns. Then one has $\det(\mathbf{A}) = 0$.

(11) Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be m different eigenvectors of an $n \times n$ matrix $\mathbf{A}$. Then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent.

(12) Let $\mathbf{A}$ be an $n \times n$ matrix such that $\mathbf{A}^2 = \mathbf{A}$. Let λ be an eigenvalue of $\mathbf{A}$. Then λ is 0 or 1.

Solution:
(1) T. Multiply A by the proposed inverse.
(2) F. Sometimes, this happens:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(3) T. The null space of BA contains the null space of A, which is non-trivial.
(4) F. A subspace should contain 0.
(5) F. One has $[x]_{\mathfrak{B}} = [2, 3, 1]^T$.
(6) T. True. More than n vectors in $\mathbf{R}^n$ are dependent.
(7) T. Look at the formula of Cramer's rule for the inverse.
(8) T. True, see book.
(9) F. The determinant is 24.
(10) T. One has $\det(A) = \det(A^T)$. One can create a zero row in A^T showing that the determinant is zero.
(11) F. If x is an eigenvector, then $x, 2x$ are dependent (it is true if the eigenvalues are different.
(12) T. If $A^2 = A$, and x is an eigenvector with eigenvalue λ, then $\lambda^2 x = A^2 x = Ax = \lambda x$. Since $x \neq 0$, we have $\lambda^2 = \lambda$, so $\lambda = 0, 1$.