Intro Linear Algebra 3A: midterm 2
Friday May 19 2017, 10:00 – 10.50 pm

There are 4 exercises, worth a total of $100 = 34 + 22 + 24 + 20$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (34 = 6 + 6 + 6 + 10 + 6 pts)
For $s \in \mathbf{R}$ consider the matrix

$$A_s = \begin{bmatrix} 1 & 2 & s \\ 1 & s+1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

(a) Compute A_s^2 when $s = 0$.
(b) Show that A_s is invertible for $s \neq 1$.
(c) For each s, compute the rank of A_s.
(d) Compute the inverse of A_s when $s = 3$.
(e) Use Cramer's rule to solve $A_s\mathbf{x} = [0, 1, 0]^T$ when $s = 0$.

Solution:
(a)

$$\begin{bmatrix} 3 & 4 & 2 \\ 3 & 5 & 2 \\ 4 & 6 & 3 \end{bmatrix}.$$

(b) $\det(A_s) = -(s-1)^2$, so invertible when $s \neq 1$.
(c) For $s \neq 1$, invertible, so rank is 3. If $s = 1$, then the rank is 1.
(d)

$$\begin{bmatrix} -\frac{1}{2} & -1 & \frac{5}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

(e) $[2, -1, 0]^T$.

Exercise 2 (22 = 7 + 6 + 3 + 6 pts)
Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 4 & 1 & 5 & 9 \end{bmatrix}$$

with reduced row echelon form

$$E = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 15 \\ 0 & 0 & 0 & 1 & 0 & -6 \end{bmatrix}$$

(a) Compute a basis of the null space of M.
(b) Compute a basis of the column space of M.
(c) Compute the dimension of the null space of M.
(d) (hard) Let N be a 6×3 matrix. Show that 6×6 matrix NM is neither onto nor one-to-one. Give a proof.

Solution:
(a) $\mathrm{Span}\{[-1, -1, 1, 0, 0, 0]^T, [-1, -2, 0, 0, 1, 0]^T, [0. - 15, 0, 6, 0, 1]^T\}$,
(b) $\{[1, 1, 3]^T, [2, 0, 1]^T, [4, 0, 1]^T\}$, pivot columns.
(c) 3
(c) M is not one-to-one, so NM is not one-to-one (free variables). N is not onto (not enough pivots), so NM is not onto.

Exercise 3 (24 = 8 + 10 + 6 pts)
Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A and show that -1 and 3 are the only eigenvalues of A.
(b) For each eigenvalue of A, compute a basis of its corresponding eigenspace.
(c) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution:
(a) $-(\lambda+1)^2(\lambda-3)$.
(b) E_{-1} has basis $\{[-1,1,0]^T, [-1/2,0,1]^T\}$. E_3 has basis $\{[1,1,0]^T\}$.
(c) Yes,

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} -1 & -1/2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Exercise 4 (20 pts)
True or false? **No** explanation required. Each question is worth 2 points.

(1) Let A, B be $n \times n$ matrices. Assume that $AB = BA$. Then $(AB)^T = A^T B^T$.
(2) Let A be an $n \times n$ matrix. Assume that there is a matrix B such that $BA = I_n$. Then A is invertible and $B = A^{-1}$.
(3) Every subspace $H \subseteq \mathbf{R}^n$ has a unique basis.
(4) Let A be an 8×11 matrix. Then $\text{rank}(A) + \dim \text{Nul}(A) = 8$.
(5) Let A be a 3×3 matrix with $\det(A) = 2^3$. Then $\det(A + A) = \det(A) \cdot \det(A)$.
(6) Let $C \subseteq \mathbf{R}^2$ be a disc with area 51. Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Then $A(C) \subseteq \mathbf{R}^2$ has area 51.
(7) Let A be an $n \times n$ matrix and assume that λ is the only eigenvalue of A and that the eigenspace at λ has dimension n. Then $A = \lambda I_n$.
(8) Let A, B be similar $n \times n$ matrices. If A is diagonalizable, then B is also diagonalizable.
(9) Let A, B be $n \times m$ matrices which have the same reduced row echelon form. Then the column spaces of A and B are the same.
(10) Let A, B be $n \times m$ matrices which have the same reduced row echelon form. Then the null spaces of A and B are the same.

Solution:
(1) True
(2) True
(3) False
(4) False
(5) True
(6) False
(7) True
(8) True
(9) False
(10) True