Intro Linear Algebra 3A: midterm 2

Wednesday February 29 2018, 3:00–3.50 pm

There are 4 exercises, worth a total of 40 points.
No calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 ($13 = 2 + 2 + 3 + 2 + 1 + 2 + 1$ pts)
For $x \in \mathbf{R}$ consider the matrix

$$A_x = \begin{bmatrix} 1 & 0 & 1 \\ 0 & x & 1 \\ 1 & 2 & x \end{bmatrix}.$$

(a) Compute $\det(A_x)$.
(b) For which x is A_x invertible?
(c) Compute the inverse of A_0.
(d) Find a basis of the null space of A_{-1}.
(e) What is the dimension of the null space of A_{-1}?
(f) Find a basis of the column space of A_2.
(g) What is the rank of A_2?

Solution:
(a) $x^2 - x - 2 = (x - 2)(x + 1)$.
(b) $x \neq 2, -1$.
(c)

$$\begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

(d) $\{[-1, 1, 1]^T\}$.
(e) 1.
(f) $\{[1, 0, 1]^T, [0, 2, 2]^T\}$.
(g) 2.

Exercise 2 ($6 = 2 + 2 + 2$ pts)
(a) Compute the matrix product

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(b) Find a matrix X such that

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} + 2X^T = 0$$

(c) Find a nonzero 2×2 matrix A with $A^2 = 0$.

Solution:
(a)

$$\begin{bmatrix} 5 & 0 & 10 \\ -1 & 0 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

(b)

$$X = -1/2 \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Exercise 3 ($11 = 2 + 1 + 4 + 2 + 2$ pts)
Consider the matrix
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 4 \end{bmatrix}.$$
(a) Compute the characteristic polynomial of A.
(b) Show that 2 and 3 are the only eigenvalues of A.
(c) For each eigenvalue, find a basis of the corresponding eigenspace.
(d) Sow that A diagonalizable by finding a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.
(e) Is A^{2018} diagonalizable? If so, find a diagonal matrix D' and an invertible matrix P' such that $A^{2018} = P'D'P'^{-1}$.

Solution:
(a) $-(\lambda - 3)(\lambda - 2)^2$.
(b) See factorization in (a).
(c) 2: $\{[2, 0, 1]^T, [0, 1, 0]^T\}$, 3: $\{[1, 0, 1]^T\}$.
(d) $D = \mathrm{diag}(2, 2, 3)$,
$$P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$
(e) Yes, take $P' = P$ and $D' = D^{2018}$.

Exercise 4 (10 pts)
True or false? **No** explanation required. Each question is worth 1 point.
(1) If A is an invertible $n \times n$ matrix and $AB = I_n$, then $B = A^{-1}$.
(2) Let A be a matrix. If the columns of A are linearly independent, then A is invertible.
(3) Let U be a subspace of $\mathbf{R}^n$. Then $\dim(U) < n$.
(4) Let A, B be 2×2 matrices with determinant 2. Then $\det(2A^T B^{-1}) = 2$.
(5) Let A be an $n \times n$ matrix with integer coefficients and determinant 1. Then A^{-1} only has integer coefficients.
(6) If A, B are $n \times n$ matrices which are both diagonalizable, then $A + B$ is diagonalizable.
(7) An $n \times n$ matrix A has eigenvalue 0 if and only if A is not invertible.
(8) If A, B are $n \times n$ matrices, then $(A - B)(A + B) = A^2 - B^2$.
(9) The points in $\mathbf{R}^3$ satisfying $x + y + z = 1$ form a subspace of $\mathbf{R}^3$.
(10) Let A be an $n \times n$ matrix with 2 identical columns. Then $\det(A) = 0$.

Solution:
(1) True.
(2) False.
(3) False.
(4) False.
(5) True.
(6) False.
(7) True.

(8) False.
(9) False.
(10) True.