Intro Linear Algebra 3A: final exam answers
Wednesday December 9, 10:30-12:30pm

Exercise 1

(a1) No, reduced row echelon form has a zero row.
(a2) No, there are free variables.
(a3) No: use either a, b or the fact that the matrix is not square.
(b1) With reduced row echelon form we find

$$\{[-1,-1,1,0,0]^T,[-2,2,0,1,0]^T,[-3,-3,0,0,1]^T\}.$$

(b2) SIze of basis, 3.
(b3) We use Gram-Schmidt with the basis from d. Note that the first 2 vectors are already orthogonal. Note that the las vector is orthogonal wrt the second vector. We compute:

$$[-3,-3,0,0,1]^T-2[-1,-1,1,0,0]^T=[-1,-1,-2,0,1]^T.$$

We find an orthogonal basis:

$$\{v_1=[-1,-1,1,0,0]^T,v_2=[-2,2,0,1,0]^T,v_3=[-1,-1,-2,0,1]^T\}.$$

Normalizing gives:

$$\{1/\sqrt{3}[-1,-1,1,0,0]^T,1/3[-2,2,0,1,0]^T,1/\sqrt{7}[-1,-1,-2,0,1]^T\}.$$

(b4) We use the orthogonal basis found before (to avoid some denominators). Note that the inner product of v with the last vector in the basis is 0. We find that the projection is equal to:

$$\frac{-3}{3}v_1+\frac{-4}{9}v_2=[1,1,-1,0,0]^T-4/9[-2,2,0,1,0]^T=[17/9,1/9,-1,-4/9,0]^T.$$

We compute v minus its projection:

$$[2,0,-1,0,0]^T-[17/9,1/9,-1,-4/9,0]^T=[1/9,-1/9,0,4/9,0]^T=1/9[1,-1,0,4,0].$$

(b5) The length of the last vector is $1/9\sqrt{18}=\sqrt{2}/3$.
(c1) First 2 columns, $\{[1,2,0]^T,[-2,1,1]^T\}$.
(c2) Size of basis in h, 2.
(c3) Read off : $2[1,2,0]^T-2[-2,1,1]^T$.
(d1) Compute a basis of $\mathrm{Nul}(A^T)$. Ore more simpler, a null space of

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

The reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 1/5 \end{bmatrix}.$$

A basis for the null space of this matrix is $\{[2,-1,5]^T\}$, and this is also a basis for $\mathrm{Col}(\mathbf{A})^\perp$.
(d2) Size of basis, 1.

Exercise 2

(a) $\det(C_a)=a^2-a-6=(a+2)(a-3)$. So not invertible if $a=-2$ or $a=3$.

(b) If $a \neq -2, 3$, the matrix is invertible and has rank 3. If $a = -2, 3$, the rank is 2.
(c) The inverse is:

$$\begin{bmatrix} -\frac{3}{2} & 2 & -1 \\ \frac{5}{4} & -\frac{3}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(d)

$$\begin{bmatrix} -2 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

(e) More solutions implies not invertible. We check $a = -2, 3$. If $a = -2$, then you see that the equation has multiple solutions if and only if $b = 4$. If $a = 3$, this happens for $b = -1$.

Exercise 3
(a) Characteristic polynomial is $(1-\lambda)^2(6-\lambda)$. Eigenvalues $1, 1, 6$.
(b) $\lambda = 1$: $\mathrm{Span}\{[0,1,0]^T, [1,0,-1]^T\}$, $\lambda 6$ gives $\mathrm{Span}\{[0,1,5]^T\}$.
(c) Yes,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 5 \end{bmatrix}.$$

and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

(d) No, the eigenspaces at 1 and 6 are not orthogonal.

Exercise 4
(a) True. One has $\det(A^{42}) = \det(A)^{42}$, which is nonzero if and only if $\det(A) \neq 0$. A matrix is invertible if and only if its determinant is nonzero.
(b) False, I_2 and $2I_2$ have the same reduced row echelon form, I_2, but they are not similary because they have different eigenvalues.
(c) True. One has $(DD^T)^2 = D(D^TD)D^T = DI_nD^T = DD^T$.
(d) True. One has

$$\lambda_1 \mathbf{v_1} \cdot \mathbf{v_2} = A\mathbf{v_1} \cdot \mathbf{v_2} = A\mathbf{v_1}^T\mathbf{v_2} = \mathbf{v_1}^T A^T \mathbf{v_2} = \mathbf{v_1}^T A \mathbf{v_2} = \mathbf{v_1} \cdot A\mathbf{v_2} = \lambda_2 \mathbf{v_1} \cdot \mathbf{v_2}.$$

Since $\lambda_1 \neq \lambda_2$, this gives $\mathbf{v_1} \cdot \mathbf{v_2} = 0$.
(e) False. If v is an eigenvector with eigenvalue λ, then one finds $\lambda v = Av = \mathbf{A}^3 v = \lambda^3 v$. So $\lambda = \lambda^3$. Hence $\lambda \neq 2$.
(f) False. Consider for example

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, G' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By looking at eigenspaces, one can see that this is not true.