Intro Linear Algebra 3A: final exam

Wednesday June 8 2016, 10:30-12:30

There are 5 exercises, worth $100 = 21 + 22 + 23 + 14 + 20$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations.

Name:

Student ID:

Exercise 1 (21 = 6 + 9 + 4 + 2 pts)
Let A be the real matrix

$$A = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A and list the eigenvalues with multiplicities.
(b) Compute a basis for each eigenspace of A.
(c) Are there a real invertible matrix P and a real diagonal matrix D such that $A = PDP^{-1}$? If yes, find such a P and D. If no, explain why not.
(d) Are there a complex invertible matrix P and a complex diagonal matrix D such that $A = PDP^{-1}$? If yes, find such a P and D. If no, explain why not.

Solution:
(a) $(\lambda-1)(\lambda-2)(\lambda-3)^2$. Eigenvalues are 1 (mult. 1), 2 (mult. 1) and 3 (mult. 2).
(b) E_1 has basis $\{[0,1,0,0]^T\}$. E_2 has basis $\{[-3,-1,1,1]^T\}$. E_3 has basis $\{[-1,0,1,0]^T\}$.
(c) No, the dimension of eigenspace at 3 is not big enough (< 2).
(d) No, the dimension of eigenspace at 3 is not big enough (< 2).

Exercise 2 ($22 = 4 + 3 + 7 + 2 + 2 + 4$ pts)
For $a \in \mathbf{R}$ set

$$A_a = \begin{bmatrix} 1 & -1 & a \\ -2 & a & 1 \\ -1 & a & -1 \end{bmatrix}$$

and set $\mathbf{v} = [0, 1, 0]^T$.
(a) Compute the determinant of A_a.
(b) For each a, determine the rank of A_a.
(c) Is A_a invertible for $a = 0$? If no, explain why not. If yes, compute its inverse.
(d) Compute a basis for the column space of A_a when $a = 1$.
(e) Compute a basis for the null space of A_a when $a = 1$.
(f) For each a determine the number of solutions to $A_a\mathbf{x} = \mathbf{v}$ (choose from $0, 1$ or ∞).

Solution:
(a) $-(a + 3)(a - 1)$.
(b) For $a \neq -3, 1$, the rank is 3. For $a = 1$ and $a = -3$, the rank is 2.
(c) Yes, its inverse is

$$\begin{bmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ -1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}.$$

(d) First two columns (from reduced row echelon form, pivot columns) $\{[1, -2, -1]^T, [-1, 1, 1]^T\}$.
(e) Basis is $\{[2, 3, 1]^T\}$.
(f) If $a \neq -3, 1$, then there is 1 solution. If $a = 1$, infinitely many. If $a = -3$, no solutions.

Exercise 3 ($23 = 4 + 1 + 7 + 2 + 5 + 4$ pts)
Let W be the following subspace of $\mathbf{R}^4$:

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

(a) Compute a basis of W.
(b) What is the dimension of W?
(c) Compute an orthogonal basis of W.
(d) Compute an orthonormal basis of W.
(e) Compute the distance from $\mathbf{v} = [0, 1, 2, 7]^T$ to W.
(f) Compute a basis of $W^\perp$.

Solution:
(a) $\{[1, 0, 1, 0]^T\}, [0, 1, 2, 0]^T, [1, 3, 1, 1]^T\}$ (the subspace is the column space of a matrix, and compute pivot columns to find basis).
(b) 3.
(c) $\{[1, 0, 1, 0]^T, [-1, 1, 1, 0]^T, [1, 2, -1, 1]^T\}$ (after applying Gram-Schmidt).
(d) $\{1/\sqrt{2}[1, 0, 1, 0]^T, 1/\sqrt{3}[-1, 1, 1, 0]^T, 1/\sqrt{7}[1, 2, -1, 1]^T\}$.
(e) $\mathbf{v} - \text{Pro}_W(\mathbf{v}) = [-1, -2, 1, 6]^T$ (use orthogonal basis computed in c). The answer is the length of this vector, which is $\sqrt{42}$.
(f) A basis is $\{[-1, -2, 1, 6]^T\}$ (one can use the answer in e).

Exercise 4 (14 = 6 + 5 + 3 pts)
Let

$$A = \begin{bmatrix} 2 & -2 \\ 6 & -5 \end{bmatrix}.$$

(a) Compute all eigenvalues of A and determine for each eigenvalue a basis of the corresponding eigenspace.
(b) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
(c) Compute A^{2016}. You can leave expressions like 7^{2016}.

Solution:
(a) Characteristic polynomial is $(\lambda+1)(\lambda+2)$. For $\lambda = -1$: $\{[2,3]^T\}$. For $\lambda = -2$: $\{[1,2]^T\}$.
(b)

$$P = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \; D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

(c)

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

One has, where $k = 2016$,

$$\begin{aligned} A^{2016} =& PD^{2016}P^{-1} = \begin{bmatrix} 2^2(-1)^k - 3(-2)^k & (-2)(-1)^k + 2(-2)^k \\ 6(-1)^k - 6(-2)^k & -3(-1)^k + 4(-2)^k \end{bmatrix} \\ =& \begin{bmatrix} 2^2 - 3(-2)^k & -2 + 2(-2)^k \\ 6 - 6(-2)^k & -3 + 4(-2)^k \end{bmatrix}. \end{aligned}$$

Exercise 5 (20 pts)
True or false? No explanation required. Points = $3 \cdot \#$ correct - 10.

(1) The matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 2i \end{bmatrix}$$

is diagonalizable over the complex numbers (where $i^2 = -1$).

(2) Let A and B be real diagonalizable matrices of the same size. Then $A + B$ is diagonalizable.

(3) Let A is an $n \times m$ matrix which is not one-to-one and let B be an $m \times s$ matrix. Then the matrix AB is not one-to-one.

(4) Consider the linear map $T : \mathbf{R}^2 \to \mathbf{R}^2$ which is a reflection through the origin. Then the standard matrix of T is $-I_2$.

(5) Let $W \subseteq \mathbf{R}^4$ be a subspace and let U be the standard matrix of the orthogonal projection Proj_W on W. Then one has $U = U^{2016}$.

(6) Let W be a subspace of $\mathbf{R}^5$. Then W has a unique orthonormal basis.

(7) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear map with matrix A. Let S be a region of $\mathbf{R}^3$ of volume 2. Then the volume of $T(S)$ is equal to $|2 \cdot \det(-A^T)|$.

(8) Let A be a real diagonalizable $n \times n$ matrix such that the characteristic polynomial can be written as $(a - \lambda)^n$ for some real number a. Then one has $A = aI_n$.

(9) Let A be an $m \times n$ matrix. Then A has a unique row echelon form.

(10) Let $\mathbf{R} \to \mathbf{R}^3$ be the map given by $x \mapsto (x^3, x^3, x^3)$. Then the range (image) of this map is a subspace of $\mathbf{R}^3$.

Solution:
(1) False. The eigenvalues are i and i, but the eigenspace has dimension 1 only.
(2) False (many counterexamples).
(3) False, take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(4) True.
(5) True.
(6) False, there are many.
(7) True.
(8) True.
(9) False (unique reduced row echelon form).
(10) True, the range is just $\{(x, x, x) : x \in \mathbf{R}\}$, which is a subspace.