Intro Linear Algebra 3A: final exam

Monday March 14 2016, 4-6 pm

There are 5 exercises, worth $100 = 14 + 23 + 23 + 20 + 20$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations.

Name:

Student ID:

Exercise 1 (14 pts)
Compute the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution:

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Exercise 2 (23= 6 + 2 + 10 + 5 pts)
Let $\mathbf{A}$ be the following real 4×4 matrix:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ -6 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of $\mathbf{A}$.
(b) Show that 1 and 2 are the only eigenvalues of $\mathbf{A}$.
(c) For each eigenvalue of $\mathbf{A}$, compute a basis of the corresponding eigenspace.
(d) Is there an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$?
If yes, find such a $\mathbf{P}$ and $\mathbf{D}$. If no, explain why not.

Solution:
(a) $(t-1)(t-2)^3$.
(b) Follows from the factorization in a.
(c) $E_1 = \text{Span}([1,0,2,0]^T)$ and $E_2 = \text{Span}([0,1,0,0]^T, [1,0,0,3]^T, [1,0,3,0]^T)$.
(d) Yes, $\dim E_1 + \dim E_2 = 4$. One can take

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

Exercise 3 ($23 = 9 + 6 + 3 + 5$ pts)
Consider the subspace W of $\mathbf{R}^4$ given by the equations $x_1 + x_2 - x_3 = 0$ and $x_1 - 2x_3 - 2x_4 = 0$. Consider the vector $\mathbf{y} = [\sqrt{6}, 0, 0, 0]^T \in \mathbf{R}^4$.
(a) Find an orthonormal basis of W.
(b) Compute the orthogonal projection $\mathrm{Proj}_W(\mathbf{y})$ of $\mathbf{y}$ on W.
(c) Compute the distance between $\mathbf{y}$ and W.
(d) Compute a basis of $W^\perp$.

Solution:
(a) The reduced row echelon form of the corresponding matrix is

$$\begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

A basis is $\{[2, -1, 1, 0]^T, [2, -2, 0, 1]^T\}$. We apply Gram-Schmidt to find the orthonormal basis $\{1/\sqrt{6}[2, -1, 1, 0]^T, 1/\sqrt{3}[0, -1, -1, 1]^T\}$.
(b) The orthogonal projection is $2/\sqrt{6}[2, -1, 1, 0]^T$ (use an orthogonal basis).
(c) The distance is the length of $1/\sqrt{6}([6, 0, 0, 0]^T - 2[2, -1, 1, 0]^T) = 1/\sqrt{6}[2, 2, -2, 0]$, which is $1/\sqrt{6}\sqrt{12} = \sqrt{2}$.
(d) One can simply take $\{[1, 1, -1, 0]^T, [1, 0, -2, -2]^T\}$ (the are the normal equations, and they are independent).

Exercise 4 $(20 = 6 + 3 + 4 + 4 + 3$ pts)
Let $c \in \mathbf{R}$ be a real number. Consider the real 3×3 matrix $\mathbf{A}_c$ given by

$$\mathbf{A}_c = \begin{bmatrix} 0 & c & 2 \\ -1 & 0 & 1 \\ c & 1 & 1 \end{bmatrix}$$

(a) Compute the determinant of $\mathbf{A}_c$.
(b) For which c is $\mathbf{A}_c$ not invertible?
(c) For $c = 1$, compute a basis for $\operatorname{Nul} \mathbf{A}_c$ and a basis for $\operatorname{Col} \mathbf{A}_c$.
(d) For $c = 0$, compute a basis for $\operatorname{Nul} \mathbf{A}_c$ and a basis for $\operatorname{Col} \mathbf{A}_c$.
(e) Compute $\mathbf{A}_0\mathbf{A}_1$.

Solution:
(a) $c^2 + c - 2 = (c-1)(c+2)$.
(b) Not invertible for $c = 1, -2$.
(c) We compute the reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the column space is (columns corresponding to pivots) is $\{[0, -1, 1]^T, [1, 0, 1]^T\}$.
A basis for the null space is given by $\{[1, -2, 1]^T\}$.
(d) The matrix is invertible, hence $\operatorname{Col} \mathbf{A}_c = \mathbf{R}^3$, and a basis is for example $\{[1, 0, 0]^T, [0, 1, 0]^T, [0, 0, 1]^T\}$. One has $\operatorname{Nul} \mathbf{A}_c = \{0\}$ and a basis is $\emptyset$ (or you can say that a basis does not exist).
(e)

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Exercise 5 (20 pts)
True or false? No explanation required. Points = $3 \cdot \#$ correct - 10.

(1) Every square matrix over the complex numbers is diagonalizable.

(2) The map $\mathbf{R}^3 \to \mathbf{R}^3$ given by $(x, y, z) \mapsto (x + y, x - z, x^2 + 2 + y - 2 - 2z - x^2)$ is not linear.

(3) Let $\mathbf{A}$ be an $n \times n$ matrix with 2 as eigenvalue. Then $\mathbf{A}^2 - 2\mathbf{A}$ is not invertible.

(4) The null space of

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \end{bmatrix}$$

is 5-dimensional.

(5) The real vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0.01 \\ 4 \end{bmatrix}$$

are linearly independent.

(6) Let $\mathbf{A}$ be an $m \times n$ matrix with linearly independent columns. Let $\mathbf{b} \in \mathbf{R}^m$. Then there is a unique vector $\mathbf{x} \in \mathbf{R}^n$ which minimizes the distance between $\mathbf{Ax}$ and $\mathbf{b}$.

(7) Let $\mathbf{A}$ be an invertible $n \times n$ matrix. Then one has $1 = \det(\mathbf{A}^{-1}\mathbf{A}^T)$.

(8) The matrix

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

represents a reflection in the line $y = x$ on the xy-plane.

(9) Let $\mathbf{A}$ be an $n \times m$ matrix and let $\mathbf{B}$ be an $m \times n$ matrix. Then one has $\dim \operatorname{Nul} \mathbf{AB} \geq \dim \operatorname{Nul} \mathbf{B}$.

(10) Let $\mathbf{A}$ be a complex $n \times n$ matrix. Assume that $\lambda \in \mathbf{C}$ is an eigenvalue of $\mathbf{A}$. Then its conjugate, $\overline{\lambda}$, is also an eigenvalue of $\mathbf{A}$.

Solution:
(1) False, for example

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not diagonalizable.
(2) False, the x^2 and 2 terms cancel out.
(3) True, for example, if x is an eigenvector at eigenvalue 2, then $(\mathbf{A}^2 - 2\mathbf{A})x = 0$, so not invertible (or one can use multiplicity of determinant).
(4) False, there are 6 free variables.
(5) True, not a multiple of each other.
(6) True, one has $\mathbf{Ax}$ is the orthogonal projection on $\operatorname{Col}\mathbf{A}$ and since the columns are linearly independent, it follows that one has a unique $\mathbf{x}$.
(7) True, $\det(A^{-1}A^T) = \det(A^{-1})\det(A^T) = \det(A)^{-1}\det(A) = 1$.
(8) False, it is the line $y = -x$.
(9) True, one even has $\operatorname{Nul} AB \supseteq \operatorname{Nul} B$.
(10) False, (true for a real matrix), take the matrix $[i]$.