Intro Linear Algebra 3A: final

Monday March 19 2018, 4:00–6.00 pm

There are 7 exercises, worth a total of 97 points.
No calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (15 = 4 + 1 + 5 + 2 + 3 pts)
Let $x \in \mathbf{R}$. Consider the matrix

$$A_x = \begin{bmatrix} 4 & x & 3 & -3 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A_x (you can leave your answer in the form $(4-\lambda)(a\lambda^3 + b\lambda^2 + c\lambda + d)$).
(b) Show that 1, 1, 4, 4 are the eigenvalues of A_x.
(c) Compute a basis for each eigenspace of A_0.
(d) Show that A_0 is diagonalizable and write $A_0 = PDP^{-1}$ where P is an invertible matrix and D is a diagonal matrix.
(e) (tricky) For which x is A_x diagonalizable?

Solution:
(a) $(\lambda - 4)^2(\lambda - 1)^2$.
(b) See a.
(c) E_4 has basis $\{[1,0,0,0]^T, [0,1,1,1]^T\}$, E_1 has basis $\{[-1,-1,1,0]^T, [1,-1,0,1]^T\}$.
(d)

$$P = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

and

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(e) Only for $x = 0$.

Exercise 2 (19 = 5 + 2 + 3 + 2 + 4 + 3 pts)
Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} \in \mathbf{R}^4.$$

Let $\mathbf{w} = [4,0,0,0]^T$. Let $U = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.
(a) Compute an orthogonal basis of U (hint: U has dimension 3 and a basis never contains a 0-vector).
(b) Compute an orthonormal basis of U.
(c) Compute the orthogonal projection of $\mathbf{w}$ on U.
(d) What is the distance from $\mathbf{w}$ to U?
(e) Find a basis for $U^\perp$.
(f) (tricky) Construct a matrix A with $\text{Nul}(A) = U$.

Solution:
(a) $\{[1,1,1,1]^T, [1,1,-1,-1]^T, [-1,1,1,-1]^T\}$.
(b) $\{1/2[1,1,1,1]^T, 1/2[1,1,-1,-1]^T, 1/2[-1,1,1,-1]^T\}$
(c) $[3,1,-1,1]^T$.
(d) 2.
(e) $\{[1,-1,1,-1]^T\}$.

(f) $[1\ -1\ 1\ -1]$.

Exercise 3 (6 pts)
Compute the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 2 & 2 \\ 1 & 3 & 0 & -1 & 0 \\ 1 & 0 & 2 & -2 & 0 \end{bmatrix}.$$

No partial credit will be given.

Solution:
60.

Exercise 4 ($19 = 3 + 6 + 6 + 4$ pts)
Consider the vectors $\mathbf{u}_1 = [1,1,1]^T$, $\mathbf{u}_2 = [1,1,0]^T$, $\mathbf{u}_3 = [1,2,3]^T$. Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Let T be the linear map with $T(\mathbf{u}_1) = \mathbf{u}_2$, $T(\mathbf{u}_2) = \mathbf{u}_3$ and $T(\mathbf{u}_3) = \mathbf{u}_1$.
(a) Show that $\mathcal{B}$ is a basis of $\mathbf{R}^3$.
(b) Consider the 3×3 matrix $B = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$. Compute B^{-1} using Cramer's rule.
(c) Compute the standard matrix of T.
(d) Compute the matrix of T in basis $\mathcal{B}$, that is, compute $[T]_{\mathcal{B}}$.

Solution:
(a) $\det(B) = 1$, so invertible and hence $\mathcal{B}$ is a basis.
(b)

$$\begin{bmatrix} 3 & -3 & 1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -4 & 7 & -3 \end{bmatrix}.$$

(d)

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Exercise 5 ($10 = 5 + 1 + 3 + 1$ pts)
Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) Compute a basis for the null space of A.
(b) What is the dimension of the null space of A?
(c) Compute a basis for the column space of A.

(d) What is the dimension of the column space of A?

Solution:
(a) $\{[1,0,0,0,0,0]^T, [0,-1,1,0,0,0]^T, [0,-3,0,-2,-1,1]^T\}$.
(b) 3
(c) $\{[1,0,0]^T, [0,1,0]^T, [0,0,1]^T\}$.
(d) 3

Exercise 6 (4 pts)
Let A be an $n \times n$ matrix such that $A^m = 0$ for some positive integer m. Show that if λ is an eigenvalue of A, then $\lambda = 0$. Give a full proof.

Solution:
If $\mathbf{x}$ is an eigenvector, with $A\mathbf{x} = \lambda\mathbf{x}$. Then $0 = A^n\mathbf{x} = \lambda^n\mathbf{x}$. So $\lambda = 0$. Hence all the eigenvalues are 0.

Exercise 7 (24 pts)
True or false? No explanation needed. Each question is worth 2 points.
(1) Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset \mathbf{R}^n$ be a set of orthogonal vectors. Then S is linearly independent.
(2) Every subspace of $\mathbf{R}^n$ has a unique orthonormal basis.
(3) Let U be an $m \times n$ matrix with orthonormal columns and let $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. Then $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \operatorname{dist}(U\mathbf{x}, U\mathbf{y})$ where dist denotes distance.
(4) Let U be a 3×3 orthogonal matrix. Then the corresponding linear map preserves volumes.
(5) Let A be a square matrix such that $A^2 + A + 3I = 0$. Then A is invertible and $A^{-1} = \frac{A^2+A}{3}$.
(6) Let A and B be similar matrices. If A is diagonalizable, then B is diagonalizable.
(7) Let A be an invertible $n \times n$ matrix and assume that $\det(A) = \det(-A)$. Then n is odd.
(8) Let A be an invertible matrix. Then $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.
(9) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear map which is the reflection in the x-axis. Let S be the linear map which is a reflection in the y-axis. Then the standard matrix of $S \circ T$ is $-I_2$.
(10) Let W be a subspace of $\mathbf{R}^n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in W$. Assume that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$ is a basis of W. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a basis of W.
(11) Let A be a 2×2 matrix and let $\mathbf{x} \in \mathbf{R}^2$. Then $||A\mathbf{x}|| = |\det(A)| \cdot ||\mathbf{x}||$.
(12) Let $\mathbf{u} \in \mathbf{R}^n$. Then the map $T : \mathbf{R}^n \to \mathbf{R}^n$ defined by $T(\mathbf{x}) = \mathbf{x} + \mathbf{u}$ is a linear map.

Solution:
(1) False.
(2) False.
(3) True.
(4) True.
(5) False.
(6) True.

(7) False.
(8) True.
(9) True.
(10) False.
(11) False.
(12) False.