Intro Linear Algebra 3A: midterm 1

Friday April 27 2018

There are 6 exercises, worth a total of 40 points.
No calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (5 pts)
Compute the inverse of

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 2 & -4 & -1 \\ -1 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix}.$$

Exercise 2 (10 = 2 + 4 + 2 + 2 pts)
Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 2 & 0 & 3 \\ 1 & 0 & -1 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Compute $A\mathbf{u}$.
(b) Compute the reduced row echelon form of the augmented matrix $[A|\mathbf{b}]$.
(c) Solve $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
(d) Show that the columns of A are linearly dependent by giving a dependence relation between the columns.

Solution:
(a) $[3, 3, 0]^T$.
(b) $\begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
(c) $[0, 1, 0, 0]^T + x_3[1, -1, 1, 0] + x_4[-1, -1/2, 0, 1]^T$.
(d) For example, column 1 - column 2 + column 3 = 0.

Exercise 3 (7 = 2 + 3 + 2 pts)
Consider the matrix and vector

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $T_A : \mathbf{R}^4 \to \mathbf{R}^4$ be the linear map which sends $\mathbf{x}$ to $A\mathbf{x}$.
(a) Compute $T_A(\mathbf{u})$.
(b) Is T_A onto? If no, find a vector which is not in the image of T_A.
(c) Is T_A one-to-one?

Solution:
(a) $[8, 6, 8, 0]^T$.
(b) No, zero row in echelon form, $[0, 0, 0, 1]^T$.
(c) No, free variables.

Exercise 4 (4 pts)
Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear map such that

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Compute the standard matrix of T.

Solution:

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}.$$

Exercise 5 (4 pts)
Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the reflection in the line $x = y$, which is a linear map. Compute the standard matrix of T.

Solution:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Exercise 6 (10 pts)
True or false? **No** explanation required. Each question is worth 1 point.
(1) If 2 augmented matrices have the same reduced row echelon form, then the solution sets of the corresponding systems coincide.
(2) There are linear systems which have precisely 3 solutions.
(3) Every matrix has a unique inverse.
(4) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbf{R}^4$ are linearly dependent, then $\mathbf{u}_1, \mathbf{u}_2$ are also linearly dependent.
(5) Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a map such that $T(\mathbf{0}) = \mathbf{0}$. Then T is a linear map.
(6) Let A be an $m \times n$ matrix and let B be an $n \times s$ matrix. Then BA is an $m \times s$ matrix.
(7) If A is an invertible matrix, then $(A^{-1})^3$ is also invertible.
(8) Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \in \mathbf{R}^5$. If $\mathbf{x} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$, then $\mathbf{x} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
(9) Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^7$. Then $2\mathbf{u}, 3\mathbf{v}, \mathbf{u} + \mathbf{v}$ are linearly dependent.
(10) Let A be an $n \times n$ matrix and let $\mathbf{v} \in \mathbf{R}^n$. Then the vectors $\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots, A^n\mathbf{v}$ are linearly dependent.

Solution:
(1) True
(2) False
(3) False
(4) False
(5) False
(6) False
(7) True
(8) False
(9) True
(10) True