Intro Linear Algebra 3A: midterm 2

Monday May 21 2018

There are 6 exercises, worth a total of 37 points.
No calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (5 pts)
Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ -2 & 3 & 1 \end{bmatrix}.$$

(a) Compute the determinant $\det(A)$ of A. (2 points)
(b) Compute A^{-1} the inverse of A using Cramer's rule (determinants). (3 points)

Solution:
(a) 6
(b)

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{3}{2} & \frac{7}{6} & -\frac{1}{6} \end{bmatrix}.$$

Exercise 2 (7 pts)
Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 \end{bmatrix}.$$

with reduced row echelon form

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis of the null space $\mathrm{Nul}(A)$ of A. (1 points)
(b) Compute the dimension of $\mathrm{Nul}(A)$. (1 point)
(c) Compute a basis of the column space $\mathrm{Col}(A)$ of A. (1 point)
(d) What is the rank of A? (1 point)
(e) Find a vector $\mathbf{b} \in \mathbf{R}^4$ with the following property: if we replace the last column of A by $\mathbf{b}$, the new matrix has rank 2. Explain your answer. (2 points)
(f) State the rank theorem. (1 point)

Solution:
(a) $\{[-1, -1, 1]^T\}$.
(b) 1 (c) $\{[1, -1, 1, 0]^T, [-1, 1, 1, 2]^T, [1, 1, 1, 1]^T\}$
(d) 3
(e) $\mathbf{b} = [0, 0, 2, 2]^T$, there will only be 2 pivots in that case.
(f) Number of columns is equal to rank plus dimension of column space.

Exercise 3 (3 pts)
Given that

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5,$$

compute

$$\det \begin{bmatrix} a & 2b & c \\ d & 2e & f \\ g & 2h & i \end{bmatrix}, \ \det \begin{bmatrix} 3a+d & 3b+e & 3c+f \\ d & e & f \\ g & h & i \end{bmatrix}, \ \det \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}.$$

No explanation required.

Solution:
10, 15, 5.

Exercise 4 (4 pts)
Let $c \in \mathbf{R}$. Consider the matrix

$$A_c = \begin{bmatrix} 1 & c & 2 \\ -1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Compute $\det(A_c)$. (2 points)
(b) For which c is A_c invertible? Explain. (1 point)
(c) If A_c is invertible, what is the rank of A_c? (1 point)

Solution:
(a) $4c - 5$
(b) $c \neq 5/4$
(c) 3

Exercise 5 (8 pts)
Let

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A. (2 points)
(b) Show that the eigenvalues of A are 1 and 2. (1 point)
(c) For each eigenvector of A compute a basis of the corresponding eigenspace. (3 points)
(d) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (2 points)

Solution:
(a) $-(\lambda - 1)(\lambda - 2)^2$.
(b) See a.
(c) E_1 has basis $\{[-1, 0, 1]^T\}$ and E_2 has basis $\{[0, 1, 0]^T, [-2, 0, 3]^T\}$.
(d) Yes,

$$D = \operatorname{diag}(1, 2, 2), \ P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

Exercise 6 (10 pts)
True or false? **No** explanation required. Each question is worth 1 point.
(1) Let A be an $m \times n$ matrix with $m > n$. Then $\dim \operatorname{Nul}(A) > 0$.
(2) Let A be a 3×3 matrix with $\det(A) = 2$. Then $\det(A \cdot A^T \cdot 2A^{-1}) = 4$.
(3) The area of the parallelogram determined by the points $(-2, -2)$, $(0, 3)$, $(4, -1)$ and $(6, 4)$ is 28.
(4) Let A be an $n \times n$ matrix. Then A is invertible if and only if 0 is not an eigenvalue of A.
(5) Let A be an $n \times n$ matrix and let $\mathbf{v}_1$, $\mathbf{v}_2$,...,$\mathbf{v}_n$ be n distinct eigenvectors of A. Then A is diagonalizable.

(6) Every upper triangular matrix is diagonalizable.
(7) Let A be an $n \times n$ matrix. Then A is invertible if and only if the columns of A form a basis of $\mathbf{R}^n$.
(8) Let A, B be $n \times n$ matrices. Then $\det(A + B) = \det(A) + \det(B)$.
(9) There is an $n \times n$ matrix with $n + 1$ different eigenvalues.
(10) Any line in $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$.

Solution:
(1) False
(2) False
(3) True
(4) True
(5) False
(6) False
(7) True
(8) False
(9) False
(10) False