Intro Linear Algebra 3A: final

Monday June 13 2018 - 2 hours

There are 7 exercises, worth a total of 90 points.
No calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (8 pts)
Consider the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}.$$

Compute A^{2018}. You can leave expressions such as $5 \cdot 3^{2019}$. (Hint: diagonalize A).

Solution:

$$A^{2018} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{2018} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 - 2^{2018} & 2 - 2^{2019} \\ -1 + 2^{2018} & -1 + 2^{2019} \end{bmatrix}.$$

Exercise 2 (15 = 4 + 4 + 3 + 2 + 2 pts)
Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A and show that the eigenvalues of A are 0 and 6.
(b) Compute a basis of each eigenspace of A.
(c) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
(d) Is $A + I$ invertible?
(e) Is $A + I$ diagonalizable?

Solution:
(a) $\lambda^2(\lambda - 6)^2$.
(b) E_0 has basis $\{[-2, 1, 0, 0]^T, [-3, 0, 1, 0]^T\}$. E_6 has basis $\{[1, 1, 1, 0]^T, [0, 0, 0, 1]^T\}$.
(c) Yes, $D = \text{diag}(0, 0, 6, 6)$ and

$$P = \begin{bmatrix} -1 & -3 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(d) Yes, since -1 is not an eigenvalue (you can plug in $\lambda = -1$ in a).
(e) Yes, Same P, but with $D = \text{diag}(1, 1, 7, 7)$.

Exercise 3 (15 = 5 + 5 + 5 pts)
Let $\mathbf{u}_1 = [1, 2, 1, -1]^T$ and $\mathbf{u}_2 = [-1, 3, 2, 0]^T$. Let $H = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\} \subseteq \mathbf{R}^4$. Let $\mathbf{y} = [0, 2, 4, 1]^T$.
(a) Compute an orthonormal basis of H.
(b) Compute the distance between $\mathbf{y}$ and H.
(c) Find a basis of $H^\perp$.

Solution:
(a) $\{1/\sqrt{7}[1, 2, 1, -1]^T, 1/\sqrt{7}[-2, 1, 1, 1]^T\}$.
(b) $\text{Proj}_H(\mathbf{y}) = [-1, 3, 2, 0]^T$ which then gives a distance $\sqrt{7}$.
(c) $\{[1, -3, 5, 0]^T, [3, 1, 0, 5]^T\}$.

Exercise 4 (18 = 2 + 1 + 4 + 2 + 2 + 2 + 1 + 2 + 2 pts)
For $x \in \mathbf{R}$ consider the matrix

$$A_x = \begin{bmatrix} 1 & 2 & 1 \\ 1 & x & 0 \\ x & 2 & 1 \end{bmatrix}.$$

(a) Compute $\det(A_x)$.
(b) Show that A_x is invertible when $x \neq 0, 1$.
(c) Compute A_x^{-1} when $x = 2$.
(d) Solve the equation $A_x\mathbf{x} = [1, 1, 0]^T$ when $x = 2$ using your answer in part (c).
(e) Solve the equation $A_x\mathbf{x} = [1, 1, 0]^T$ when $x = 0$ in parametric vector form.
(f) For which x is $\{[1, 0, 0]^T, [0, 1, 0]^T, [0, 0, 1]^T\}$ a basis of the column space of A_x?
(g) Compute the rank of A_x for all x.
(h) Compute a basis of the null space of A_x when $x = 0$ and when $x = 1$.
(i) Compute the dimension of the null space of A_x for all x.

Solution:
(a) $x(x - 1)$.
(b) See a.
(c) $\begin{bmatrix} -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 0 \end{bmatrix}$.
(d) $\{[-1, 1, 0]^T\}$.
(e) $x_3[0, -1/2, 1]^T + [1, 0, 0]^T$.
(f) For $x \neq 0, 1$.
(g) If $x \neq 0, 1$ it is 3. For $x = 0, 1$ it is 2.
(h) When $x = 0$, $\{[0, -1, 2]^T\}$. When $x = 1$, $\{[1, -1, 1]^T\}$.
(i) If $x \neq 0, 1$, it is 0. For $x = 0, 1$ it is 1.

Exercise 5 ($8 = 2 + 2 + 2 + 2$ pts)
Let $S : \mathbf{R}^2 \to \mathbf{R}^2$ be the reflection in the x-axis. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation of 180 degrees around the origin (clockwise). You may use that S and T are linear maps.
(a) Find the standard matrix of S.
(b) Find the standard matrix of T.
(c) Find the standard matrix of $S \circ T$.
(d) Give a nice/simple description of the map $S \circ T$.

Solution:
(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
(b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
(c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
(d) Reflection in y-axis.

Exercise 6 ($6 = 3 + 3$ pts)
Let A be an $n \times n$ matrix such that $A^2 = A$.
(a) Show that $0, 1$ are the only possible eigenvalues of A.
(b) Show that any $\mathbf{y} \in \mathbf{R}^n$ can be written as $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ with $\mathbf{y}_1 \in \mathrm{Nul}(A)$ and $\mathbf{y}_2 \in \mathrm{Col}(A)$. (Hint: $\mathbf{y} = (\mathbf{y} - A\mathbf{y}) + A\mathbf{y}$).

Solution:
(a) One has $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then one finds $\lambda\mathbf{x} = A\mathbf{x} = A^2\mathbf{x} = \lambda^2\mathbf{x}$. This gives $\lambda^2 = \lambda$.
(b) Set $\mathbf{y}_1 = \mathbf{y} - A\mathbf{y}$ and $\mathbf{y}_2 = A\mathbf{y}$.

Exercise 7 (20 pts)
True or false? No explanation needed. Each question is worth 2 points.
(1) Let A be an $n \times n$ matrix. If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are eigenvectors of A, then so is $\mathbf{x} + \mathbf{y}$.
(2) Let U be an orthogonal matrix. Then $\det(U) = 1$ or $\det(U) = -1$.
(3) Let $c \in \mathbf{R}$ and let I be the $n \times n$ identity matrix. Then $\det(cI) = c$.
(4) Let A be an $n \times n$ matrix. Then every vector $\mathbf{y} \in \mathbf{R}^n$ can be written uniquely as $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ with $\mathbf{y}_1 \in \mathrm{Nul}(A)$ and $\mathbf{y}_2 \in \mathrm{Col}(A)$.
(5) Every square matrix is diagonalizable over the complex numbers.
(6) Let A, B be nonzero $n \times n$ matrices. Then there is an $n \times n$ matrix C with $A = BC$.
(7) Let H be a subspace of $\mathbf{R}^n$. Then H has an orthonormal basis.
(8) Let A be an 5×10 matrix. Then $\dim(\mathrm{Nul}(A)) \geq 4$.
(9) Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$. Then $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cdot \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
(10) Let A be an $m \times n$ matrix. Then A has a unique reduced row echelon form.

Solution:
(1) F
(2) T
(3) F
(4) F
(5) F
(6) F
(7) T
(8) T
(9) F
(10) T