Intro Linear Algebra 3A: midterm 1

Wednesday April 20, 2:00- 2:50pm

There are 3 exercises, worth a total of $100 = 42 + 28 + 30$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (42 = 6 + 16 + 8 + 5 + 7 pts)
Let

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ -3 & 0 & -1 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

(a) Compute AC.
(b) Compute the reduced row echelon form of the augmented matrix $[A|\mathbf{b}]$.
(c) Solve $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
(d) Is the linear map corresponding to A onto?
(e) Are the columns of A linearly independent? If they are dependent, find a linear dependence relation among the columns of A.

Solution:
(a)

$$\begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -4 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{bmatrix}.$$

(c) $x_4[0, -1, 0, 1, 0]^T + x_5[1, 0, -2, 0, 1]^T + [0, 2, 3, 0, 0]^T$.
(d) Yes, pivot in every row of reduced row echelon form of A (no zero rows in RREF of A).
(e) No. For examle, second column minus fourth column is 0 (find any vector in null space: you can find null space from c, which is just $x_4[0, -1, 0, 1, 0]^T + x_5[1, 0, -2, 0, 1]^T$).

Exercise 2 ($28 = 7 + 15 + 6$ pts)
Consider the vectors

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbf{R}^3$.
(a) Show that $\mathbf{e_1} = [1,0,0]^T$, $\mathbf{e_2} = [0,1,0]^T$ and $\mathbf{e_3} = [0,0,1]^T$ are in the span of $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$.
(b) Construct a 2×3 matrix A such that $A\mathbf{u_1} = [0,1]^T$, $A\mathbf{u_2} = [1,0]^T$ and $A\mathbf{u_3} = [1,1]^T$.
(c) How many linear maps $T : \mathbf{R}^3 \to \mathbf{R}^2$ satisfy $T(\mathbf{u_1}) = [0,1]^T$, $T(\mathbf{u_2}) = [1,0]^T$ (choose from 0, 1, or infinitely many)? Explain.

Solution:
(a) One can compute the RREF of $[\mathbf{u_1 u_2 u_3}]$, which is the identity: this means that any augmented system is consistent. In fact, one has:

$$\begin{aligned} \mathbf{e_1} &= -\mathbf{u_2} + \mathbf{u_3} \\ \mathbf{e_2} &= -\mathbf{u_1} + 3\mathbf{u_2} - \mathbf{u_3} \\ \mathbf{e_3} &= 2\mathbf{u_2} - \mathbf{u_3}. \end{aligned}$$

(these are unique solutions).
(b) There are multiple ways of solving this, one is by using 6 variables for the matrix system, and then solving it. But, in our case, we can use our expressions from (a) to find:

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & -2 & -1 \end{bmatrix}.$$

This matrix satisfies the required properties, and is unique.
(c) Infinitely many. If one makes a system, one gets a system in 6 variables with 4 equations which is consistent by (b). Hence there are at least 2 free variables and infinitely many solutions.

Exercise 3 (30 pts)
True or false? **No** explanation required. Points is $-10 + 4 \cdot \#\text{correct}$.

(1) The vectors $[1, 2, 3]^T$ and $[2, 4, 5]^T$ are linearly dependent.

(2) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there is a unique $n \times m$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^n$.

(3) The map $\mathbf{R}^3 \to \mathbf{R}^3$ given by $(x, y, z) \mapsto (xy + 1 - yx - 1 + x + y, y + z, z)$ is not linear.

(4) Let T be the linear map on the xy-plane which is the reflection in the x-axis. Then T is one-to-one.

(5) Let A be an $n \times n$ matrix and let $\mathbf{x} \in \mathbf{R}^n$ be such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbf{R}$. Then one has $A^{42}\mathbf{x} = \lambda^{42}\mathbf{x}$.

(6) Let A be a matrix with n columns, p pivots and f free variables. Then one has $n = p + f$.

(7) Let $T : \mathbf{R}^7 \to \mathbf{R}^3$ be a linear map. Then T is onto.

(8) There is a linear map $T : \mathbf{R}^2 \to \mathbf{R}^2$ with

$$T([1, 0]^T) = [1, 0]^T,\ T([0, 1]^T) = [1, 1]^T,\ T([1, -1]^T) = [0, 1]^T.$$

(9) The linear map corresponding to the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is the projection onto the x_1-axis.

(10) Let A, B be square matrices such that $AB = BA$. Then one has $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

Solution:
(1) False, not multiple of each other.
(2) False, it is an $m \times n$ matrix.
(3) False, it is linear, just $(x, y, z) \mapsto (x + y, y + z, z)$.
(4) True (if $T\mathbf{x} = T\mathbf{y}$, then $\mathbf{x} = T^2\mathbf{x} = T^2\mathbf{y} = \mathbf{y}$).

(5) True, $A^{42}\mathbf{x} = \lambda A^{41}\mathbf{x} = \ldots = \lambda^{42}\mathbf{x}$.
(6) True, one sees this from the definition.
(7) False, 0 matrix is not onto.
(8) False, the first two equations show that the matrix would be

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

But this matrix has $A[1, -1]^T = [0, -1]^T$, which is not $[0, 1]^T$. So no such map exists.
(9) False, the map is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(10) True, one can just work out the produt and use $AB = BA$.