Intro Linear Algebra 3A: midterm 1

Friday January 29, 3:00- 3:50pm

There are 3 exercises, worth a total of $100 = 48 + 16 + 36$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations.

Name:

Student ID:

Exercise 1 (48 = 6 + 22 + 8 + 4 + 4 + 4 pts)
Let

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 1 & 4 \\ 1 & 0 & 1 & 1 & 8 \\ -3 & 0 & -1 & 0 & -5 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 4 \\ 10 \\ -8 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Compute $\mathbf{Ac}$.
(b) Compute the reduced row echelon form of $[\mathbf{A}|\mathbf{b}]$.
(c) Solve $\mathbf{Ax} = \mathbf{b}$ in parametric vector form.
(d) Solve $\mathbf{Ax} = \mathbf{0}$ in parametric vector form.
(e) Is $\mathbf{A} : \mathbf{R}^5 \to \mathbf{R}^3$ onto?
(f) Are the columns of $\mathbf{A}$ linearly independent?

Answers:
(a) $[-5, -5, 0]^T$.
(b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 5 & 6 \end{bmatrix}$$

(c)$x_2[0, 1, 0, 0, 0, 0]^T + x_5[-1, 0, -2, -5, 1]^T + [2, 0, 2, 6, 0]^T$.
(d) $x_2[0, 1, 0, 0, 0, 0]^T + x_5[-1, 0, -2, -5, 1]^T$: just take the part with free variables.
(e) Yes, every row has a pivot.
(f) No, there are free variables.

Exercise 2 ($16 = 6 + 6 + 4$ pts)
Consider the $xy-$plane $\mathbf{R}^2$. Let T_1 be the linear map which rotates 90 degrees clockwise around the origin. Let T_2 be the linear map which reflects in the y-axis.
(a) Compute the matrix $\mathbf{A}$ of the linear map T_1.
(b) Compute the matrix $\mathbf{B}$ of the linear map T_2.
(c) Compute $\mathbf{AB}$ and $\mathbf{BA}$.

Answers:
(a)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(b)

$$\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c)

$$\mathbf{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathbf{BA} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Exercise 3 ($36 = 3 + 3 + \ldots + 3$ pts)
True or false? **No** explanation required.

(1) The matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is in reduced row echelon form.
(FALSE): upper right one should not be there.

(2) The matrix $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ has precisely 1 free variable.
(FALSE): thre are 2 free variables. (3) The map $\mathbf{R}^2 \to \mathbf{R}^2$ defined by $(x, y) \mapsto (x, \sin y)$ is not linear.
(TRUE): (proof omitted).
(4) The map $\mathbf{R}^2 \to \mathbf{R}^2$ defined by $(x, y) \mapsto (x + y, 2x + y + 1)$ is linear.
(FALSE): 0 does not go to 0.
(5) The linear map given by the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ is not one-to-one.
(TRUE) It has free variables, so not one-to-one (3 columns in $\mathbf{R}^2$ are linearly dependent).
(6) The vectors $\{[1, 2, 3]^T, [2, 3, 4]^T, [0, 0, 0]^T\}$ are linearly independent.
(FALSE): dependent because of 0 vector.
(7) If $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{A}$ is one-to-one, then so is $\mathbf{A}^2$.
(TRUE): If $A^2x = 0$, then $A(Ax) = 0$. Since A is one-to-one this means $Ax = 0$. Hence $x = 0$.
(8) The set $\{[1, 2]^T, [1, 3]^T\}$ spans $\mathbf{R}^2$.
(TRUE): reduced row echelon form is identity matrix.
(9) There is no 4×4 matrix $\mathbf{A}$ such that the map $\mathbf{A} : \mathbf{R}^4 \to \mathbf{R}^4$ is one-to-one but not onto.
(TRUE) one-to-one means no free variables, so every column has a pivot, so every row has a pivot.
(10) Let $\mathbf{A}, \mathbf{B}$ be 2×2 matrices. Then one has $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.
(FALSE) this is true if and only if $\mathbf{AB} = \mathbf{BA}$.
(11) Let $\mathbf{A}, \mathbf{B}$ be 2×2 matrices such that $\mathbf{AB} = \mathbf{BA}$. Then one has $\mathbf{ABABAAB} = \mathbf{BABAABA}$.
(TRUE): Both are A^4B^3.
(12) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a function (a map). Then there is a 2×2 matrix $\mathbf{A}$ such that for all $\mathbf{x} \in \mathbf{R}^2$ one has $T(\mathbf{x}) = \mathbf{Ax}$.
(FALSE): This is only true if T is linear.