Intro Linear Algebra 3A: midterm 1

Monday January 30 2017, 00:00 – 00.50 pm

There are 3 exercises, worth a total of $100 = 42 + 28 + 30$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 ($42 = 6 + 16 + 7 + 3 + 5 + 5$ pts)
Let

$$A = \begin{bmatrix} 4 & -1 & 7 & -7 \\ 1 & 3 & 5 & 21 \\ 2 & 0 & 4 & 0 \\ 1 & 1 & 3 & 7 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Compute $A\mathbf{b}$.
(b) Compute the reduced row echelon form of the augmented matrix $[A|\mathbf{b}]$.
(c) Solve $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
(d) Solve $A\mathbf{x} = \mathbf{0}$ in parametric vector form.
(e) Is the linear map corresponding to A one-to-one?
(f) Is the span of the columns of A equal to $\mathbf{R}^4$?

Solution:
(a) $[14, -29, 2, -9]^T$;
(b)

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 7 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(c)

$$x_3[-2, -1, 1, 0]^T + x_4[0, -7, 0, 1]^T + [0, -1, 0, 0]^T.$$

(d)

$$x_3[-2, -1, 1, 0]^T + x_4[0, -7, 0, 1]^T.$$

(e) No, free variables.
(f) No, zero rows in RREF.

Exercise 2 (28 = 5 + 5 + 12 + 6 pts)
Consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

in $\mathbf{R}^2$. Furthermore, consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

in $\mathbf{R}^3$.
(a) Show that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent and find a linear dependence relation between them.
(b) Show that the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ span $\mathbf{R}^2$.
(c) Construct a matrix A with $A\mathbf{u}_1 = \mathbf{v}_1$, $A\mathbf{u}_2 = \mathbf{v}_2$ and $A\mathbf{u}_3 = \mathbf{v}_3$.
(d) Is there a matrix A with $A\mathbf{u}_1 = \mathbf{v}_1$, $A\mathbf{u}_2 = \mathbf{v}_2$ and $A\mathbf{u}_3 = 2\mathbf{v}_3$?

Solution:
(a) $2\mathbf{u}_1 - \mathbf{u}_2 - \mathbf{u}_3 = \mathbf{0}$ (or a nonzero multiple of this relation).
(b) Recuded row echelon form of $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ has no zero rows.
(c)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(d) No, you get an inconsistent system, since $A2\mathbf{u}_1 - A\mathbf{u}_2 - A\mathbf{u}_3 \neq A\mathbf{0} = \mathbf{0}$.

Exercise 3 (30 pts)
True or false? **No** explanation required. Points is $-10 + 4 \cdot \#\text{correct}$.

(1) Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be a map such that $T(\mathbf{0}) = \mathbf{0}$ and such that for $c \in \mathbf{R}$ and $\mathbf{u} \in \mathbf{R}^3$ one has $T(c\mathbf{u}) = cT(\mathbf{u})$. Then T is a linear map.
(2) Let m be a positive integer and let $T : \mathbf{R}^m \to \mathbf{R}^m$ be a linear map which is one-to-one. Then T is onto.
(3) Let A be an $m \times n$ matrix. Then A has a unique row echelon form.
(4) Let A be a matrix. Then the equation $A\mathbf{x} = \mathbf{0}$ has either precisely 1 or infinitely many solutions.
(5) Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^4$ be linearly dependent vectors. Then $\mathbf{u}$ is a multiple of $\mathbf{v}$.
(6) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear map which is reflection in the line $x_1 = x_2$. Then the standard matrix of this map is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(7) The map $T : \mathbf{R}^2 \to \mathbf{R}^3$, $(x, y) \mapsto (x + y, 1 + x - y - y - 1, y + x + y)$ is a linear map.
(8) Let $[A|\mathbf{b}]$ be an augmented matrix. If we apply a row operation to this matrix, then the solution set might change.
(9) Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$. Then the vectors $\mathbf{u}, \mathbf{v}, 2\mathbf{u} + 3\mathbf{v}$ are linearly dependent.
(10) Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}.$$

Then $\mathbf{w}$ is in the span of $\mathbf{u}, \mathbf{v}$.

Solution:
(1) False.
(2) True.
(3) False (unique reduced row echelon form).
(4) True.
(5) False (zero vector can be in there).
(6) True.
(7) True.
(8) False.
(9) True.
(10) True.