Intro Linear Algebra 3A: midterm 1

Friday April 21 2017, 00:00 – 00.50 pm

There are 4 exercises, worth a total of $100 = 36 + 36 + 20 + 8$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (36 = 6 + 12 + 6 + 4 + 4 + 4 pts)
Let

$$A = \begin{bmatrix} 3 & 2 & -1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & -1 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}.$$

(a) Compute $A\mathbf{c}$.
(b) Compute the reduced row echelon form of the augmented matrix $[A|\mathbf{b}]$.
(c) Solve $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
(d) Is there a $\mathbf{b}' \in \mathbf{R}^3$ such that the equation $A\mathbf{x} = \mathbf{b}'$ has a unique solution?
(e) Is the linear map corresponding to A one-to-one?
(f) Is the span of the columns of A equal to $\mathbf{R}^3$?

Solution:
(a) $[-1, -4, -1]^T$;
(b)

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{bmatrix}$$

(c)

$$[1, 1, 1, 0, 0]^t + x_4[1, -2, 2, 1, 0]^T + x_5[1, -1, 1, 0, 1]^T.$$

(d) No, always infinitely many - RREF will always be consistent and have free variables.
(e) No, free variables.
(f) Yes, no zero rows in RREF.

Exercise 2 ($36 = 4 + 4 + 5 + 10 + 10 + 3$ pts)
For $c \in \mathbf{R}$ consider

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$$

(a) Show that $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent.
(b) Describe in words what the span of $\mathbf{u}_1$ and $\mathbf{u}_2$ looks like as a subset of $\mathbf{R}^3$.
(c) Are $\mathbf{u}_1, \mathbf{u}_2$ and $42\mathbf{u}_1 - 1296\mathbf{u}_2$ linearly independent?
(d) For which c are $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ linearly dependent? For each such c, find a dependence relation.
(e) Construct a matrix A such that $A\mathbf{u}_1 = \mathbf{u}_2$ and $A\mathbf{u}_2 = \mathbf{u}_1$.
(f) Is the answer to (e) unique?

Solution:
(a) The vectors are not multiple of each other, so independent.
(b) It is a plane through the origin.
(c) $-42\mathbf{u}_1 + 1296\mathbf{u}_2 + (42\mathbf{u}_1 - 1296\mathbf{u}_2) = 0$, so dependent.
(d) For $c = 1/2$. One has $-3[1, 1, 1]^T + [1, 2, 3]^T + 2[1, 1/2, 0]^T = 0$.
(e) For example, one can take

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{bmatrix}.$$

(f) No, there are many options (free variables).

Exercise 3 (20 pts)
True or false? **No** explanation required. Each question is worth 2 points.
(1) Every matrix has more than 1 row echelon form.
(2) Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbf{R}^5$. Let $\mathbf{b} \in \mathbf{R}^5$ with $\mathbf{b} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Then $\mathbf{b} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
(3) The standard matrix of the reflection in the line $x_2 = x_1$ on $\mathbf{R}^2$ is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(4) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with standard matrix A. Then T is one-to-one if and only if the columns of A are linearly independent.
(5) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^7$. Then one has $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{w} + \mathbf{v})$.
(6) The vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix}$$

are linearly dependent.
(7) Assume that the matrix equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. Then the equation $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.
(8) A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.
(9) The augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{array}\right].$$

has 4 free variables.
(10) The map

$$\begin{aligned} \mathbf{R}^3 &\to \mathbf{R}^3 \\ (x, y, z) &\mapsto (3y + x, 2x + y + z, 3x + 2 + z) \end{aligned}$$

is linear.

Solution:
(1) False
(2) True
(3) False
(4) True
(5) True
(6) True
(7) True
(8) True
(9) False, only 3.
(10) False

Exercise 4 ($8 = 4 + 4$ pts)
Prove the following statements.
(a) Matrices with the same reduced row echelon form can be obtained from each other by using row operations.
(b) Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map. If T is onto, then T is one-to-one.

Solution:
(a) The row operations are invertible (check row replacement). Hence one can transform matrix A to RREF, and then transform it to B by inverting the operations which make B into its reduced row echelon form.
(b) The corresponding matrix A has no zero rows, and because it is square, has no free variables. Hence the map is one-to-one.