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# REMARKS ON REFLECTION PRINCIPLES, LARGE CARDINALS, AND ELEMENTARY EMBEDDINGS[1]

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This paper is intended as a brief introduction to recent work directed at the formulation of new axioms of infinity or large cardinal properties. It is intended to be as free from technicalities as possible, and aspires to explain the motivating ideas as clearly and fully as possible, with attention to the problems, puzzles, and intuitions that lie behind them. We include some comments on developments since the 1967 symposium.

A paper to be titled *Strong axioms of infinity and elementary embeddings* is in preparation with R. M. Solovay, and has been owed for some time now. The present paper may serve as a first installment on this debt. (Solovay is not, however, to be blamed for any deficiencies or excesses of this exposition.) The full payment will include a more complete discussion of the relation between large cardinal-embedding conditions and ultrafilter-measure conditions, and also of the known consequences of axioms we have considered.

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[1] A significant part of this paper was written while the author was visiting the University of Amsterdam. I am especially indebted to Professor M. H. Löb for a number of discussions concerning the problem of existence in mathematics.

I have also benefited from discussions with Professor K. Gödel. However, as to the justification of the axioms attempted here, Gödel feels that it is rather unsatisfactory and that something better could be achieved by a proper analysis of the notion of *structural property* of the concept of set, which would then lead to the reflection principles in the form "any structural property of the concept of set is reflected by some set." What is done here may be viewed as a proposal that the structural properties of properties of sets are those of the form $jP$ for some property of sets $P$. This, of course, makes the concept of structural property relative to the particular representation of "imaginary situations," and is thus at best an approximation to Gödel's idea. As for extendibility itself, Gödel feels it is mathematically natural, but is about as well justified by the observation that it asserts an analogy between $\kappa$ and $\lambda > \kappa$, which is plausible in the way that compactness of $\omega$ suggests the existence of a strongly compact cardinal.

**1. Reflection principles.**

1.1. *Paradoxes in logic; Cantor's Ω and reflection principles.* It has been pointed out that the paradoxes of Russell, Burali-Forti never really caused a crisis in mathematics (where one deals only with unproblematic examples of sets), but rather in logic (and general set theory) where one attempts to provide a general and universal framework for mathematics and in particular for arbitrary sets. We now consider such a frame to have been provided for set theory by the clarification of the intuitive idea of the cumulative hierarchy (due chiefly to Zermelo) (i.e., the sets in the series $R_0, R_1, \ldots$ given by $R_{\alpha+1} = \mathscr{P}R_\alpha$ carried into the transfinite for arbitrary ordinals $\alpha$ by the rule $R_\lambda = \bigcup_{\alpha<\lambda} R_\alpha$ in case $\lambda$ is a limit ordinal; here $\mathscr{P}X = \{t | t \subseteq X\}$.) We might say that this reduces all structural questions to questions about "arbitrary subset of" and "arbitrary ordinal." (The apparent restriction on universality arising from sets whose elements are not sets and "irregular" sets is not believed to be significant for the expressive power of the framework. The reason is that *every* set $X$ is supposed to be equivalent to an ordinal, and since we are concerned only with structural questions (i.e., up to isomorphism) it suffices to consider the corresponding ordinal. This is independent of the possible interpretational or epistemic interest of such sets.) The picture provided suffices to set up the basic axioms of set theory. It rather explicitly refrains from a definite division of "ordinals" from other existing objects, and, in fact, does not tell us much about the transfinite sequence of ordinals Ω (which Cantor conceived as "absolutely infinite"). Insofar as we know anything more about this, our knowledge seems to depend on so-called reflection principles. There are concrete examples here which seem quite unproblematic. (Exactly what is included is perhaps debatable, but we would certainly include inaccessible cardinals, Mahlo cardinals, and indescribable cardinals. If our analysis is correct, then measurable, supercompact, and extendable cardinals are to be included, as well as Vopěnka's principle. The latter however requires a further development than that given here.) The difficulties arise in formulating a general principle. For example, analogous to the "ideal" comprehension principle of naive logic, we have the naive reflection principle:

(P1) If Ω has any property $P$ then there is an ordinal $\kappa < \Omega$ which also has the property $P$.

If we consider the property $P(x) \leftrightarrow x = \Omega$, we see that something more subtle is required. However, this form brings out two aspects of reflection principles clearly. There is (i) some consideration of ("reflection on") the universe of all sets or on Ω and its properties, and (ii) there is claimed to exist an object (here, a set $\kappa$) which mirrors ("reflects") this universe. The difficulties lie in assigning suitable meanings to "Ω has $P$" (which generally seems to involve consideration of objects which are not sets) while preserving in some way the universality of set theory.

1.2. *Examples of reflection arguments.* It may be helpful to give some informal arguments illustrating the use of reflection principles.

The simplest is perhaps: the universe of sets is inaccessible (i.e., satisfies the replacement axiom), *therefore* there is an inaccessible cardinal. This can be elaborated somewhat, as follows. Let $\theta_\nu$ enumerate the inaccessible cardinals. By the same sort of reasoning, $\theta_\nu$ is not bounded: the Cantor absolute $\Omega$ (all ordinals) is an inaccessible above any proposed bound $\beta$, *therefore* there is an inaccessible cardinal above $\beta$. Clearly, then, there are $\Omega$ inaccessibles below $\Omega$; *therefore* there is an inaccessible $\kappa$ such that there are $\kappa$ inaccessibles below it (i.e., $\kappa = \theta_\kappa$). Now suppose that $F$ is a set function $F = \Omega \to \Omega$ such that it is nondecreasing and continuous (i.e., at limits $F\lambda = \bigcup \{F\alpha | \alpha < \lambda\}$). Now it is natural to think of $F\Omega$ as making sense (we can "define" it by the continuity condition) in which case it is clear that $F\Omega = \Omega$. Thus $\Omega$ is an inaccessible fixed point of $F$. If we allow reflection on *this* statement, we get that $F$ has an inaccessible fixed point. Notice that we could have thought of the argument above for $\kappa$ similarly: $\theta_\Omega = \Omega \to (\exists\kappa < \Omega)\theta_\kappa = \kappa$.

We give one more example. A *tree* is a partial ordering in which each initial segment is well-ordered. The rank of $x$ is the order type of $\{y | y < x\}$. Let $T$ be any tree structure on $\Omega$ such that (i) for each $\alpha < \Omega$ the points of $T$ of rank $< \alpha$ form a set (rather than a proper class), and (ii) $T$ has points of arbitrarily high rank. We argue that $T$ must have a branch of length $\Omega$. First choose $F_\alpha$ to be the tree obtained from $T$ by keeping only points of rank $< \alpha$. Now $F_\alpha$ is a set, and, for $\alpha < \beta$, $F_\beta$ is an end extension of $F_\alpha$. For limits $\lambda$, $F_\lambda = \bigcup\{F_\alpha | \alpha < \lambda\}$. Notice that each $F_\alpha$ has a branch of length $\alpha$ (since $F_{\alpha+1} \sim F_\alpha \neq 0$). Now let us apply $F$ formally to $\Omega$. Then formally we expect $F_\Omega$ to have a branch of length $\Omega$. And, indeed, if we suppose $F_\Omega$ does *not* have such a branch, and apply reflection, we get a contradiction. Finally, $F_\Omega = \bigcup \{F_\alpha | \alpha < \Omega\} = T$, so $T$ has such a branch.

At this point we can say that our method of making such informal arguments precise is (i) to make precise the sense in which a function $F : \Omega \to \Omega$ can be applied to $\Omega$, and (ii) to make precise the statements about $F$ for which reflection is allowed.

1.3. *An imprecise remark.* It is possible to give a metaphysical motivation for the form of (P1) along Cantorian lines. We present this in the hope that some readers will find it illuminating. According to Cantor, $\Omega$ is unlimited.[2] To the extent that our thinking is limited, then, it should be compatible with what we think or understand (about $\Omega$ for example) that the same could be thought or understood of some $\kappa < \Omega$. Thus we do not understand (in the requisite sense) the property $P(x) \leftrightarrow x = \Omega$.

A similar doctrine can be formulated about the totality of all things which exist: whatever theory we have about what exists, it should be compatible with our understanding of our theory that the totality of existing things should be a set. This places a restriction on the expressive power of "understandable" languages. (In the case of pure-unapplied-first-order logic, the doctrine is proved by the Löwenheim-Skolem theorem.)

**2. $\Omega$ from outside.** As a first attempt at understanding the mechanics of reflection arguments, let us imagine for a moment that we could get outside of Cantor's

[2] More precisely, Cantor conceived of the *process of formation* of the ordinals as "*absolut* grenzenlos" (letter to Dedekind). Cantor shared with classical Greece a certain distaste for the *ἄπειρον* (unbounded, indeterminate), under which he classified the potential infinite.

universe $V = R_\Omega$, and think of ("reflect on") $V$ as if it were a set. Then there would be objects (such as $\Omega$, $\Omega + 1$, $R_{\Omega+\Omega}$) of rank $\geqq \Omega$. This of course already violates part of our intuition about $\Omega$, namely, that it is all *possible* ordinals (or well-order types), but aside from this there seems nothing inconsistent in such an imaginative foray. It may be compared to the "virtual displacements" which play a role in the extremum principles of physics. Formally, it is very easy to implement (the difficulties come in interpreting the formalism). We add to the language of ZF individual constants $V, \Omega$ together with the usual axioms relativized to $V$. Also we add an axiom asserting that $V = R_\Omega$. (This last assures that for $x \in V$, $\mathscr{P}x$ contains all possible subsets of x.) The question arises: What theory do we assume in dealing with the new objects we have projected? Now Cantor's universe $V$ is intended to comprehend all possibilities as regards well-orderings; so it is natural to suppose that the theory of $R_\Omega$ would be applicable in our projected situation. If we allow the complete theory (sets as parameters) we obtain by this analogical procedure the schema

$$(\forall x, y \in V)[\theta^V(x, y) \leftrightarrow \theta(x, y)]. \tag{S2}$$

(Here $\theta$ is any $\in$-formula whose free variables are $x$, $y$, and $\theta^V$ is obtained from $\theta$ by relativizing all quantifiers to $V$. Syntactically, $V$ is an individual constant.) The schema simply asserts that any first-order sentence $\theta^V$ of the theory of $R_\Omega$ is true in the projected universe.

2.1. *Reflection arguments from the projection schema.* We indicate how (S2) may be used to formalize reflection principle arguments. Now the formula of set theory which expresses "x is inaccessible" can be formally applied to $V$. In this sense we can, within this formalism, assert that $V$ is inaccessible (which of course is not expressible by a first-order formula of the language of ZF when the quantifiers range over $V$). Let us add the axiom "$V$ is inaccessible" to our formalism. It is now possible, in those cases where the set function $F$ is definable, to carry out the informal arguments given earlier. Utilizing informal criteria to judge when the passage from "for all definable $F$" to "for all $F$" is acceptable, one can in this way obtain all the Mahlo cardinals. (The informal step is in each case analogous to the passage from "$V$ satisfies replacement for definable functions" to "$V$ is inaccessible.") For details, see Lévy [**L1**], [**L2**]. The schema (S2) can be used to provide an elegant axiomatization of set theory. The resulting theory, which we will call ZA (because of the similarity to Ackermann's set theory [**A1**]) is equivalent to ZF + (S2) + $V = R_\Omega$, and its $\in$-part is ZF. (See [**R1**], [**R3**], [**L1**].)

**3. The language of ZF vs. language with $\Omega$: a digression on theories, methods of extending theories, and properties of natural models.** The reader has probably noticed that the schema (S2) corresponds to the following relation $E(\kappa; \lambda)$ on ordinals.

DEFINITION 3.1. Let $\kappa$, $\lambda$ be ordinals. We write $E(\kappa; \lambda)$, and say that $\kappa$ is first-order extendible to $\lambda$, in case

$$(R_\kappa, \in) \prec (R_\lambda, \in) \;\&\; \kappa < \lambda. \tag{3.1}$$

Why not dispense with formalisms involving nonsets (or "imagined but nonexisting" sets) and discuss properties of sets like the above? This is usually desirable (especially for convenience in precise formulation), but our concern here is with motivation and evidence, which, like it or not, seem to depend on consideration of $\Omega$. Moreover, the schema (S2) is supposed to assert something about correct theories of sets and correct methods of extending such theories, which is not contained for example in the bare assertion that there are $\kappa$, $\lambda$ such that $E(\kappa;\lambda)$. We stop here to discuss this because it helps explain the role that $\Omega$ plays in building set theories.

In devising set theories we are interested in

(a) theories $T$ which are correct on the intended interpretation,

(b) procedures $T \to T'$ which always lead from correct theories to correct theories.

Now "correctness of $T$ on the intended interpretation" is supposed to be expressible in the language of (S2); that is what $\Omega$ or $V$ is for. Thus, letting $\psi(T, V)$ be a formula expressing "$T$ is true in $V$," whenever

(a1) $\mathrm{ZA} \vdash \psi(T, V)$,

(b1) $\mathrm{ZA} \vdash \forall T[\psi(T, V) \to \psi(T', V)]$,

(where $T \mapsto T'$ is a function expressible in ZF) then we have corresponding principles of type (a), (b) above. Thus uses of ZA to get either type of assertion are seen to be special cases of

$$[\mathrm{ZA} \vdash \varphi(V)] \to \varphi(V)$$

which is just the "proof theoretic reflection principle" for ZA (written in a sloppy manner, however). Notice that common methods of extending theories are contained in this; for example $T \mapsto T +$ "$T$ is consistent," $T \mapsto T +$ "$T$ has a natural model," etc.[3]

3.1. *Large cardinal properties.* Within the language of ZA we can also say (for a single $\in$-formula $\varphi$) what it means for $\varphi$ to express a large cardinal property. It means that $\varphi(\Omega)$. Now one can obtain principles of types (a), (b) from a large cardinal property and a given correct theory of sets (such as ZF) as follows. If $\varphi$ is a large cardinal property

(a2) $\mathrm{ZF} \vdash \forall\kappa(\varphi(\kappa) \to \psi(T, R_\kappa))$,

(b2) $\mathrm{ZF} \vdash \forall\kappa(\varphi(\kappa) \to \forall T[\psi(T, R_\kappa) \to \psi(T', R_\kappa)])$,

then we have the corresponding principles of type (a), (b) above. (Again, $\psi(T, X)$ expresses "$T$ is true in $X$.")

Thus in giving a model theoretic condition (instead of a formal theory) one must specify what is intended as the large cardinal. For example, in (3.1), $\kappa$ is

[3] The connection between such reflection principles and set theoretic reflection principles does not seem to me to be merely verbal. Here one considers what is true, and this is mirrored by what is provable. There is also the element of ostensible reflexivity: one's considerations are turned back upon themselves (e.g. one tries to be conscious of the formal system one is using). It seems, however, that they always fall back to something less than themselves. This element also occurs in set theoretic reflection principles, but the ostensible reflexivity is more ontological: we reflect on the (mathematical) existence of that which we consider, as we consider mathematical existence.

intended as the large cardinal. Since in (S2) we consider not only statements of the form $\psi(T, V)$ ($T$ is correct) but also $\varphi(V)$ ($\varphi$ is a large cardinal property), we get from a condition such as (3.1) (and a correct theory such as ZF) versions of (a), (b) for large cardinal properties. Let $\theta(\kappa, \lambda)$ formalize (3.1). If

(a3) $\mathrm{ZF} \vdash \theta(\kappa, \lambda) \to \varphi_0^{R_\lambda}(\kappa)$,

(b3) $\mathrm{ZF} \vdash \theta(\kappa, \lambda) \wedge \varphi^{R_\lambda}(\kappa) \to \bar{\varphi}^{R_\lambda}(\kappa)$,

then $\varphi_0$ is a large cardinal property and $\varphi \mapsto \bar{\varphi}$ is an operation which preserves large cardinal properties.

3.2. *Formal consistency*. One further point can be made here. The theory ZA (ZF + (S2) + $V = R_\Omega$) is provably consistent relative to ZF (see Lévy [**L1**]). But the proof of this does not appear to provide an interpretation of ZA which gives $\Omega$ the role which we are suggesing here. In fact, the formal consistency of ZA merely asserts the existence of models $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{B}$, etc., and is thus not as strong as $\exists\kappa, \lambda E(\kappa, \lambda)$, which we have already observed does not capture the role of $\Omega$.[4]

**4. Bernays' reflection principle using classes**. The rather formalistic approach via (S2) which we have taken to reflection principles has the advantage that (ostensibly at least) we are never required to accept the existence of objects other than sets, or of relations other than $\in$. Everything else is merely a formal device. The appropriateness of the device must be judged informally. In its defense it may be pointed out that, assuming we want to think of $V$ at all, the analogical procedure (think about it as you would think about anything else) seems to have an edge until either (a) it runs into trouble or proves fruitless or (b) some informal considerations suggest an alternative to classical logic or classical set theory which has some chance of proving fruitful in this context.

One disadvantage is that we would like a better account of the passage from "all definable $F$" to "all $F$." Allowing proper classes to exist would seem to be a fair price to pay for this, especially since proper classes seem needed anyway just to *express* the properties we are concerned with (such as inaccessibility). We therefore consider means of expressing reflection principles in the language of class-set theory (GB or KM). (Since we now have objects which are not sets, the universality of set theory is threatened. We shall return to this point.)

[4] Some further remarks may help clarify the situation. If $N$ is a model of GB, then $(V^N, \in^N)$, or $V^N$ for short, is a model of ZF. Similarly for ZA. If $C$ is a class of models of the language for ZF, write $T \vdash_C \sigma$ iff $\forall N(N \models T \,\&\, V^N \in C \to V^N \models \sigma)$. Since *any* model $M$ of ZF can be extended to a model $N$ of GB having the same sets, for all $C$ we have $\mathrm{ZF} \vdash_C \sigma$ iff $\mathrm{GB} \vdash_C \sigma$, and we may regard ZF and GB as essentially the same theory of sets. The new objects of GB can be regarded as a mere *facon de parler*. (Of course, one need not so regard them, and indeed Gödel did not regard them this way in his monograph introducing $L$; but this is beside the point.) Although any model $M$ of ZF gives rise via Lévy's proof to a model $N^*$ of ZA, $N^*$ is not in general an extension of $M$ having the same sets ($M$ and $(V^*, \in^*)$ need not be isomorphic). Thus $V$ does not capture the notion of set of the original interpretation $M$ and thus does not play the role we desire. In fact, the equivalence $\mathrm{ZF} \vdash_C \sigma$ iff $\mathrm{ZA} \vdash_C \sigma$ fails if $C$ is the class of natural models $R_\kappa$ (ZA is the stronger theory here). Nevertheless, Lévy's proof shows that $\mathrm{ZF} \vdash \sigma$ iff $\mathrm{ZA} \vdash \sigma^V$ iff $\mathrm{ZA} \vdash \sigma$, so that (in a weaker sense than for GB) ZA and ZF are essentially the same theory, and in this weaker sense the "imaginary" objects can be regarded as a *facon de parler*.

Let us suppose that $\kappa < \Omega$ is going to reflect $\Omega$. Then sets correspond to elements of $R_\kappa$, and proper classes to subsets of $R_\kappa$. Thus if we wish to allow proper classes $F \subseteq R_\Omega$ as parameters in reflectable statements, it is quite natural to suppose that what is true of $F$ will be true of $F \cap R_\kappa$ in the reflecting universe. This suggests the schema

$$\text{(S3)} \qquad \theta(F, G) \to (\exists\kappa < \Omega)\theta^{R_{\kappa+1}}(F \cap R_\kappa, G \cap R_\kappa)$$

(since statements $\theta$ are really of the form $\theta^{R_{\Omega+1}}$ in class-set theory). Note that this is close to the form of (P1).

4.1. *An interpretation eliminating imaginary sets in favor of sets.* Using (S3) one can immediately give an interpretation of (S2) which is closer to (P1). Let $F$ be (or code) the Skolem functions for $R_\Omega$. Then there is a $\theta$ which says $R_\Omega$ is closed under the Skolem functions. Thus choosing $\kappa$ for this $\theta$, $R_\kappa$ is closed also, i.e.,

$$\text{(4.1)} \qquad R_\kappa \prec R_\Omega.$$

(Evidently one can conjoin other properties of $\Omega$ with $\theta$ before using (S3) to get $\kappa$. Those familiar with KM will note that it will suffice to get the unembellished (4.1).) Now take the quantifiers of (S2) to range over $R_\Omega$, and take $V$ to be $R_\kappa$. Note that in this interpretation no quantifiers go outside $R_\Omega$. However, the definition or proof of existence of $\kappa$ appears to require going beyond $R_\Omega$.

4.2. *Eliminating imaginary sets in favor of classes.* Still another interpretation of (S2) exists on the basis of (S3). We can show, in effect, that there is a transitive "set" $M$ such that

$$\text{(4.2)} \qquad (R_\Omega, \in) \prec (M, \in).$$

Actually we can only get $M$ up to isomorphism; the exact statement is: There exists a well-founded relation $E \subseteq R_\Omega$ (say with field $M$), a point $\bar{\Omega}$ in $M$, and an isomorphism $j$ such that

$$(R_\Omega, \in) \cong_j (R^M_{\bar{\Omega}}, E) \prec (M, E).$$

The proof (by contradiction) is easy. If it fails, choose an inaccessible $\kappa$ reflecting the statement that it fails, and such that $R_\kappa \prec R_\Omega$. By the downward Löwenheim-Skolem theorem, there is a transitive set $M$ of cardinality $\kappa$ such that $R_\kappa \prec M$, a contradiction.

We remark that these interpretations are in a sense nonstandard: In the first, the "large cardinal" $\kappa$ may not have all large cardinal properties it should; in the second the properties of $\Omega$ may not relativize properly to $M$. However, in either case the interpretation is adequate for expressing first-order properties of $R_\Omega$.

4.3. *Reflection arguments using Bernays' schema.* Using (S3), it is possible to carry out all the informal arguments given earlier (and even prove the replacement axiom). For details see Bernays [**B1**], where (S3) is introduced as the basis of a very elegant axiomatization of class-set theory. (By taking care in the relativization of equality, Bernays even avoids the necessity of relativizing the parameters $F$ to $F \cap R_\kappa$.)

**5. Imaginary sets and classes; the set-class distinction.** We may regard Bernays' schema as a reflection principle for second-order logic over set theory. There are obvious generalizations to higher orders; these are the so-called indescribability conditions. (Bernays' schema asserts that $\Omega$ is second-order indescribable. The notion of indescribability is discussed in Lévy [L3].) Passing to higher types gives stronger axioms. None of these generalizations, however, allows objects of type higher than classes of sets as parameters in the reflected statements. Since it is the presence of the classes as parameters in Bernays' schema which appears to be responsible for its great strength, we would eventually like to continue by formulating a third-order reflection principle which allows third-order objects as parameters. It does not appear to be possible to generalize (S3) in this direction very directly. (The reader who does not believe this is welcome to try.) We therefore return to the considerations that led to (S2), and ask whether we can formulate a version of (S2) appropriate to class-set theory.

5.1. *The set-class distinction; a projection schema.* In class-set theory we have classes as distinct from sets; but if we conceive of them merely as collections, this looks like a distinction without a difference. In particular, we seem to have nothing not contained already in (S2) plus "$V$ is inaccessible." Moreover, the classes threaten the universality of set theory. (This perhaps is why many mathematicians find ZF far more natural than KM; our idea of set comes from the cumulative hierarchy, so if you are going to add a layer at the top it looks like you just forgot to finish the hierarchy.) A proper class $P$ may however be distinguished from a set $x$ in the following way (if the reader will indulge another counterfactual conditional): If there were more ordinals (or if, as in (S2), $\Omega$ were an ordinal), $x$ would have exactly the same members, whereas $P$ would necessarily have new elements. We could say that the extension of $x$ is fixed but that of $P$ depends on what sets exist. Roughly, $x$ *is* its extension, whereas $P$ has more to it than that.[5] Notice that in (S2) the definable properties behave in just this way; the extension of $\theta$ is $P = \{x \in V | \theta(x)\}$, whereas in the projected universe it becomes $\{x | \theta(x)\}$. (Thus $\theta$ determines a set just in case the two are the same—which agrees with the "size" criterion which is frequently used to motivate the axioms of ZF.) This tells us how to formulate the analogue of (S2) for class-set theory: Write $jP$ for the extension of $P$ in the formally projected universe; then the schema we want is

$$\text{(S4)} \qquad (\forall x, y \in V)(\forall P \subseteq V)[\theta^{\mathscr{P}V}(x, y, P) \leftrightarrow \theta(x, y, jP)].$$

Syntactically $j$ is a unary function symbol. We allow it in the comprehension principle appropriate to class-set theory:

$$\exists z \forall t[t \in z \leftrightarrow \theta \wedge \exists u(t \in u)].$$

Otherwise (S4) is just like (S2). ($x, y, P$ exhaust the free variables of $\theta$; $V$ is a new individual constant; $\theta^{\mathscr{P}V}$ indicates relativization of all quantifiers to "$x \subseteq V$"; we suppose the usual axioms for $V$ and that $V$ has the form $R_\Omega$.)

[5] We could say that $P$ contains the famous three dots or "etc." of mathematics in an essential way.

Note that if $x \in V$, also $x \subseteq V$, so that $x = x \leftrightarrow x = jx$ is an instance of (S4). This assures that $j$ is the identity on $V$.

In (S2), the original universe $V$ consists of sets; in the formal projection we treat $V$ as a set, thus projecting new sets. In (S4), the original universe consists of sets and classes of sets. In the formal projection we introduce new sets and classes of *these* as well; among these are the $jP$.

Schema (S4) asserts that sentences of the theory of sets, allowing quantifiers over classes, and both sets and classes as parameters, are true in the projected universe when the class parameter $X$ is given its proper extension $jX$.

5.2. *Model theoretic version.* The model theoretic condition corresponding to (S4) is $E^1(\kappa; \lambda)$:

DEFINITION 5.1. We write $E^1(\kappa; \lambda)$ and say that $\kappa$ is second-order extendible to $\lambda$ in case

$$\kappa < \lambda \text{ and } \exists M, j: R_{\kappa+1} \to M \subseteq R_{\lambda+1} \text{ such that}$$

$$\text{(i) } (M, \in) \prec (R_{\lambda+1}, \in), \tag{5.1}$$

$$\text{(ii) for all } x \in R_\kappa, jx = x.$$

Evidently $j\kappa = \lambda$; we call $\kappa$ the critical point of the embedding $j$ (since it is the first point moved by $j$). It is $\kappa$ which is intended as the large cardinal. In the sequel we write (5.1) in the more abbreviated form

$$j: R_{\kappa+1} \cong \prec R_{\lambda+1},$$

understanding that $\kappa$ is the critical point (and hence $\kappa < \lambda$).

Note that this condition can also be expressed by saying that the complete theory of the structure $(R_{\kappa+1}, \in, x, X)_{x \in R_\kappa, X \in R_{\kappa+1}}$ has an interpretation $(R_{\lambda+1}, \in, x, X')_{x \in R_\kappa, X \in R_{\kappa+1}}$ in which $\kappa$ is a set ($\kappa < \lambda$).

5.3. *Universality of set theory: applicability to classes.* In conceiving classes $P \subseteq V$ we must decide whether their extensions are to be treated in a perfectly classical way, or not; whether, for example, the axiom of choice is to hold. If set theory is as universal as we intend, it should be applicable to any collections, including these. We note that (S4) decides strongly in favor of such a classical treatment. The reason for this is that while classes $P \subseteq V$ are treated as having "potential extensions" $jP$, they (or, if you prefer, their extensions) are simultaneously treated formally as sets. In the formally projected universe, $V$ is a set, so we have in effect identified projected subsets of $V$ with the extensions of our proper classes. Since the theory of $V$ is supposed to apply in the projected universe, $V = R_\Omega$ and $R_{\Omega+1}$ are treated classically. In this way we mitigate the threat to the universality of set theory which is posed by the introduction of proper classes.

5.4. *Virtues of the class-set projection schema.* So far we have introduced (S4) only as a formal device. (We prefer to give it some interpretation. In the sequel we will suggest, for example, interpretations analogous to the interpretations (4.1), (4.2) which Bernays' schema (S3) provides for the first-order case (S2).) As such it has some noteworthy virtues. (i) It works extremely smoothly in formalizing the earlier informal arguments (it is only necessary to read $(jF)\Omega$ for $F\Omega$). (ii) Bernays' schema (S3) follows easily from it. (iii) As we shall see, it is very easy to generalize it to allow higher-order parameters. (iv) It allows us to make an intelligible conceptual distinction between sets and proper classes.

5.5. *References*. The condition (S4) (or its model theoretic version) was first formulated by Silver following a suggestion of Reinhardt. It is closely related to the set theory of Ackermann, to informal ideas of Shoenfield, and to a set theory of Powell (for details, see Jech-Powell [**J1**], Reinhardt [**R1**], and Shoenfield [**S1**, p. 234]). The condition was used by Silver [**S2**] in his proof of the relative consistency of "$2^\kappa \neq \kappa^+$ and $\kappa$ measurable."

5.6. *Free historical remark*. Formal devices such as $j$ and the idea of analogical predication are hardly new to the history of thought. The author recently stumbled on an early use of a $j$-like device by the writer known as Pseudo-Dionysius (fifth century(?)); Greek scholars can probably find earlier antecedents. According to the "negative way" of Proclus as followed by Pseudo-Dionysius, created beings are (for example) wise or unwise (as the case may be), but the Creator is neither; he is superwise. According to (S4), of course, ordinals are accessible or not as the case may be, but $\Omega$ is $j$ (inaccessible). See Coppleston [**C1**, pp. 51–52; also compare pp. 61–62].

**6. Higher type objects ($\Omega$-classes); representing imaginary sets and classes using classes ($\Omega$-classes).** We wish to indicate how (S4) may be interpreted. The chief difficulty here is explaining what kind of object the quantifiers range over. Before assuming there are objects satisfying the axioms we would like at least to know what sort of things they are going to be. Of course we can suppose there is a set which is a model, but this defeats the intention that the $R_\Omega$ part of the theory really applies to Cantor's universe and that proper classes are really classes of sets. On the other hand, if we introduce new objects (even proper classes) beyond Cantor's universe this violates the universality of the concept of set. We propose to mitigate this sorrow by seeing the universality not in the *extension* of the concept of set but in the applicability of the *theory* of sets. (We have already seen how this works in the case of classes in observing how (S4) treats classes classically.)

6.1. *$\Omega$-classes*. The simplest way to proceed is perhaps the following. We have now introduced proper classes of sets as distinct from sets. (To be sure, the distinction is drawn only by considering "imaginary" sets and classes.) In a similar way we can consider classes of proper classes, classes of these, etc. We call these $\Omega$-classes, since they are built up over $\Omega$. The manner in which proper classes are not "purely extensional" is indicated by (S4). In an analogous manner, the $\Omega$-classes are conceived as "nonextensional"; this will be spelled out below. They may, however, also be considered purely extensionally.[6] Since we regard set theory (the theory of $\Omega$) as *the* theory of extensional objects such as sets, collections, etc.,

[6] That is, without projecting any imaginary universe. The essential nonextensionality appears only when this is done.

we assume this theory applies to $\Omega$-classes. This is expressed by the schema (S2), which we adopt explicitly here.

AXIOM 6.1. *We adopt the schema* (S2) *interpreting* "$\forall x$" *as "for all* $\Omega$*-classes* $x$," $\Omega$ *as the Cantor series of ordinals, and* "$V$" *as* $R_\Omega$.

Of course, this assumption gives us an enormously rich supply of $\Omega$-classes.

In order to express the distinctive characteristics of $\Omega$-classes we could now introduce imaginary sets and imaginary $\Omega$-classes. However, we wish to avoid as much as possible an unending series of extensions of the types of objects allowed in our theories (especially "imaginary" objects). Therefore we shall use the $\Omega$-classes we already have, making free use of the natural models $R_\beta$ ($\Omega < \beta$) which are available, to represent imaginary objects and to explain the special axioms for $\Omega$-classes. Thus while our set-class distinction does presuppose the idea of a set-possible set distinction, we do avoid the introduction of possible sets. This still makes liberal use of $\Omega$-classes, so we shall examine how many are needed for certain purposes; but at first we ignore such niceties.

6.2. *The special axiom for* $\Omega$*-classes.* Let $\lambda > \Omega$, so that $R_\lambda$ is a collection of $\Omega$-classes. We wish to consider a realm $R_{\Omega'}$ of imaginary sets, and a corresponding realm $R_{\lambda'}$ of imaginary $\Omega$-classes, in which $\Omega$ is an imaginary set. Moreover we want classes of sets to correspond to classes of imaginary sets, and other $\Omega$-classes $x$ to correspond appropriately to other imaginary classes $jx$ in such a way that the theory is preserved. In the case $\lambda = \Omega + 1$, this preservation is expressed by the satisfaction of (S4); in model theoretic terms for $R_\lambda$, $R_{\Omega'}$, $R_{\lambda'}$, this says that there exists $j: R_\lambda \to R_{\lambda'}$, such that

(6a) (i) $\Omega < j\Omega = \Omega' < \lambda'$,

(ii) $\forall x < \Omega, jx = x$,

(iii) if $M = \{jx | x \in R_\lambda\}$, $(M, \in) \prec (R_{\lambda'}, \in)$.

In addition, in the case $\lambda = \Omega + 1$ we automatically have

(6b) (iv) $\lambda < \Omega'$

which, of course, says that the $\Omega$-classes considered are in the realm of imaginary sets. It is (6) which we use to indicate the special character of the $\Omega$-classes.

We put these conditions in a formal definition.

DEFINITION 6.2. Let $\Omega, \lambda, \Omega', \lambda'$ be ordinals, $j$ a function $j: R_\lambda \to R_{\lambda'}$.

(a) We write $j: E_0(\Omega, \lambda; \Omega', \lambda')$ in case $\lambda > \Omega$ and the conditions (6a)(i)–(iii) above are satisfied. We call $\Omega$ the critical point of the embedding $j$. In case the condition (6b)(iv) above is also satisfied, we write $j: E(\Omega, \lambda; \Omega', \lambda')$.

(b) If there is such a $j$ we write simply $E_0(\Omega, \lambda; \Omega', \lambda')$ or $E(\Omega, \lambda; \Omega', \lambda')$ and say that $\Omega$ is $\lambda$-extendible (to $\Omega'$).

(c) We say that $\Omega$ is extendible if it is $\lambda$-extendible for every $\lambda > \Omega$.

We make explicit our special assumption about $\Omega$-classes.

AXIOM 6.3. *For every* $\Omega$*-class* $\lambda$ *which is formally an ordinal,* $\Omega$ *is* $\lambda$*-extendible.*

Several comments are needed about (6). (a) Although it is obvious, we note that it is $\Omega$ that is the large cardinal here and not $\lambda$. Of course if one thinks of $R_\lambda$ as all

$\Omega$-classes, or all existing classes, or the like, it seems reasonable to expect that properties which hold of $\lambda$ will be large cardinal properties; but this does require an additional conceptual step. (b) The condition (i) says that $\Omega$ is an imaginary set, or is imagined as a set, (ii) that sets are allowed as parameters without application of $j$ (as in schemas (S2) and (S4)), and (iii) that truth is preserved in the projected universe. The condition (iv) does not seem implicit in the idea of projecting imaginary sets and $\Omega$-classes. However, if we want to treat not only $\Omega$ but also the $\Omega$-classes in $R_\lambda$ as sets, it is clearly necessary that $\lambda \leq \Omega'$. Thus by taking an $\Omega'$ for $\lambda + 1$ we get $\lambda < \lambda + 1 \leq \Omega'$, so that (iv) is also seen to be a natural condition here.

6.3. *$\Omega$-classes = possible $\Omega$-classes*? Having introduced $\Omega$-classes, we may ask whether it makes sense to consider all possible $\Omega$-classes. Since we already have so much difficulty with all possible *sets*, one may expect at least analogous difficulties here. However, if we do think of $\lambda$ as representing all possible $\Omega$-classes, it seems fairly clear that $\lambda = \lambda'$. (As $\lambda'$ cannot be smaller than $\lambda$, it has nowhere else to go if $\lambda$ is all *possible* $\Omega$-classes. In other words $\lambda$ remains the same in the imaginative projection while $\Omega$ changes to $\Omega'$.) This is clearly incompatible with (iv) as we surely must have $\Omega' < \lambda'$ ($\Omega < x < \lambda$ therefore $\Omega' = j\Omega < jx < \lambda'$); but we observed that (iv) did not seem forced by our guiding idea. If we now retain the conditions (6a), we are led to the conclusion that $(R_\lambda, \in)$ has a proper elementary embedding into itself. In 1967 it seemed plausible to the author that such an absolute treatment of $\lambda$ should be possible, and hence that there should be inaccessible cardinals $\lambda$ of this sort. Such an axiom was proposed in the author's thesis [**R2**]. This however was a mistake: Kunen has proved (using the axiom of choice) that any such ordinal $\lambda$ must either be of the form $\beta$ or $\beta + 1$ where $\beta$ has cofinality $\omega$ [**K1**]. Whether the cofinality condition is necessary and whether the AC is inessential or alternatively the condition provides an interesting way of violating AC (as with the axiom of determinateness) remain open problems.

The above line of thought which suggests a proper elementary embedding of $R_\lambda$ into itself can be set in a slightly different context. There is a natural tendency to try to keep everything within the realm of sets, and to regard all talk of $\Omega$ as actually about an ordinal $\kappa$ which reflects $\Omega$ to a suitable degree. (We of course have opted to allow $\Omega$-classes and to secure the universality of set theory in another way.) If one does this it is natural to consider $\kappa$ which are $\lambda$-extendible for every ordinal $\lambda$. But in this context (keeping everything within the realm of sets), the next step, namely $\kappa$ which are $\infty$-extendible (meaning one $j$ which works for all $\lambda$) is just like setting $\Omega$-classes = imaginary $\Omega$-classes above and yields a proper elementary embedding of the universe of sets into itself. This was the approach of [**R2**], where it was insisted that the contemplation of alternatives to the actual universe of sets (here accomplished by the distinction set-imaginary set, and represented using $\Omega$-classes) be in terms of representations of $V$ which are sets.

6.4. *An alternative motivation.* There is an alternative motivation for elementary embedding conditions which may also suggest considering the condition $V \cong \prec V$ (proper). It is well known (from Scott [**S3**]) that if $\kappa$ is a measurable cardinal, then there is an elementary embedding $j$ of the universe of regular sets $V$ into a well-founded class of sets $M$, with $\kappa$ the first ordinal moved by $j$. Moreover, as stronger closure conditions are imposed on $M$ ($M$ closed under sequences of length $2^\kappa$, $2^{2^\kappa}$, etc.), the assertions become arithmetically stronger (actually, $\kappa$ increases in size). This suggests various elementary embedding conditions including $V \cong \prec V$.

While this line of thought is motivated by the desire for stronger assertions and by a natural generalization, and does motivate some of the conditions we introduced earlier, it is not the original motivation. (Kunen [**K1**] presents it in such a way that the reader might think it was.) Rather, it is very close to the motivation for considering irreducible covers (see § 7). Moreover as this presupposes the idea of a measure, and does not seem to have much direct connection with the idea of reflection or other basic ideas, it seems to have quite different foundational relevance than the approach of §§ 6.2, 6.3.

6.5. *A priori possibilities for $\Omega$-classes.* If $R_\lambda$ are $\Omega$-classes, and $\Omega'$, $R_{\lambda'}$ represent respectively the imaginary ordinals and the imaginary $\Omega$-classes corresponding to $R_\lambda$, then there are four a priori possibilities for the order relations among $\Omega, \lambda, \Omega', \lambda'$. Clearly $\Omega < \Omega' < \lambda'$, and $\Omega < \lambda \leq \lambda'$. Thus the possibilities are given by the placement of $\lambda : \lambda < \Omega'$, $\lambda = \Omega'$, $\Omega' < \lambda < \lambda'$, and $\lambda = \lambda'$. The first and last have been considered above. We note here the four conditions are of increasing strength.

THEOREM 6.4. *Let $\Omega, \lambda_0, \ldots$ be any ordinals. The conditions*

(i) $j : E_0(\Omega, \lambda_0 ; \Omega', \lambda_0') \;\&\; \lambda_0 = \lambda_0'$,

(ii) $j : E_0(\Omega, \lambda_1 ; \Omega', \lambda_1') \;\&\; \Omega' < \lambda_1 < \lambda_1'$,

(iii) $j : E_0(\Omega, \lambda_2 ; \Omega', \lambda_2') \;\&\; \lambda_2 = \Omega'$,

*are of decreasing strength:*

(a) *if* (i) *holds,* $\&\; \lambda_1 = j\Omega'$, $\lambda_1' = j\lambda_1$, *then* (ii) *holds, and*

(b) *if* (ii) *holds,* $\&\; \lambda_2 = \Omega'$, $\lambda_2' = j\lambda_2$, *then* (iii) *holds.*

*Moreover, if* (iii) *holds, then in the universe* $R_{\lambda_2}$, $\Omega$ *is extendible, i.e.,* $\forall \lambda < \lambda_2$, $\exists \Omega', \lambda' < \lambda_2$ *such that*

(iv) $E_0(\Omega, \lambda ; \Omega', \lambda')$ *and* $\lambda < \Omega'$.

PROOF. The first part is obvious. For the second, suppose (iii) holds. We note that $R_\Omega \prec R_{\Omega'}$, i.e., that, in $R_{\Omega'} = R_\lambda$,

$$(\forall x \in R_\Omega)(\theta^{R_\Omega}(x) \leftrightarrow \theta(x)),$$

which is a statement of the form $\varphi(\Omega)$. Now by the embedding of $R_\lambda$ into $R_{\lambda'}$, this means that in $R_{\lambda'}$, $\varphi(j\Omega)$, i.e., that $\varphi(\lambda)$ or

$$(\forall x \in R_\lambda)(\theta^{R_\lambda}(x) \leftrightarrow \theta(x))$$

which means that $R_{\lambda'} \prec R_\lambda$. Since, for any $\beta < \Omega'$, $\Omega$ is clearly $\beta$-extendible in $R_{\lambda'}$, it must also be so in $R_\lambda$ as desired.

**7. Measures, supercompactness**. We now introduce the notion of irreducible cover. This notion arises naturally when one investigates $\kappa$-complete ultrafilters

(0-1 measures). It can be regarded as a generalization of the notion of normal measure. (A $\kappa$-complete measure $\mu$ on $\kappa$ is said to be normal in case (i) if $\nu < \kappa$, $\mu\{x \in \kappa | \nu < x\} = 1$, (ii) if $f : \kappa \to \kappa$ and $\mu\{x | fx \in x\} = 1$, then there is $\nu \in \kappa$ such that $\mu\{x | fx = \nu\} = 1$. Every measurable cardinal has a normal measure.) A supercompact cardinal is one admitting an irreducible cover of every set. The terminology and this form of the definition are due to Solovay, the notion independently to Solovay and the author. In § 7, Axioms 6.1, 6.3 are not used.

DEFINITION 7.1. (a) $\mu \subseteq \mathscr{P}S$ is called an ultrafilter on $S$ in case for all $X, Y \subseteq S$,

(i) $X \in \mu$ or $S \sim X \in \mu$,

(ii) if $X \in \mu$ and $X \subseteq Y$, then $Y \in \mu$,

(iii) if $X, Y \in \mu$ then $X \cap Y \in \mu$.

(b) An ultrafilter $\mu$ is called $\kappa$-complete if whenever $\lambda < \kappa$ and for all $\nu < \lambda$, $X_\nu \in \mu$, also $\bigcap\{X_\nu | \nu < \lambda\} \in \mu$.

(c) A $\kappa$-complete ultrafilter $\mu$ is said to be of degree $\kappa$ if it is not $\kappa^+$ complete.

DEFINITION 7.2. Suppose $\mu$ is an ultrafilter of degree $\kappa$ on $S$. Then

(a) A function $f : S \to \mathscr{P}A$ is said to be a cover of $A$ (of degree $\kappa$) in case

(i) for each $a \in A$, $\{x \in S | a \in fx\} \in \mu$,

(ii) card $fx < \kappa$ for each $x \in S$.

(b) A cover of $A$ is said to be irreducible if in addition

(iii) whenever $g$ is a function on $S$ and $\{x \in S | gx \in fx\} \in \mu$, there is an $a \in A$ such that $\{x \in S | gx = a\} \in \mu$.

DEFINITION 7.3. (a) A cardinal $\kappa$ is said to be strongly compact in case every set has a cover of degree $\kappa$.

(b) A cardinal $\kappa$ is said to be supercompact in case every set has an irreducible cover of degree $\kappa$.

We begin by observing that every cover of $A$ of degree $\kappa$ determines a cover whose space is $\mathscr{P}_\kappa A = \{H \subseteq A \mid \text{card } H < \kappa\}$ and whose covering function is the identity. Namely, if $\mu'$ is the measure on $S$, define $\mu$ on $\mathscr{P}_\kappa A$ by $X \in \mu$ iff $\{x \in S | fx \in X\} \in \mu'$. The degree of $\mu$ is also $\kappa$, and if $\mu'$ is an irreducible cover so is $\mu$.

THEORM 7.3. *If $\kappa$ is extendible then $\kappa$ is supercompact.*

PROOF. Let $A$ be given; we will get an irreducible cover for $A$. Choose $\lambda > \kappa$ so that $A \in R_\lambda$. For convenience we also suppose $\lambda$ a limit ordinal. By extendibility, choose $j, \lambda'$ so $j : R_\lambda \cong R_{\lambda'}$ with $\kappa$ as critical point. We now induce an ultrafilter $\mu$ on $\mathscr{P}_\kappa A$ by

$$X \in \mu \quad \text{iff} \quad j''A \in jX$$

where $j''A = \{ja | a \in A\}$. Using the identity function, this provides the required cover of degree $\kappa$.

We omit the easy proof that $\mu$ is an ultrafilter. To see it is $\kappa$-complete, let $X_\nu \in \mu$, $\nu < \lambda < \kappa$. Now let $Y = \{X_\nu | \nu < \lambda\}$. Since $\lambda < \kappa$, $j\lambda = \lambda$, so $jY = \{(jX)_\nu | \nu < \lambda\}$. Similarly $j(X_\nu) = (jX)(j\nu) = (jX)_\nu$, so $jY = \{j(X_\nu) | \nu < \lambda\}$. Evidently $j''A \in jY$ since it is in each $j(X_\nu)$.

To see that $\mu$ is a cover, we must see that for each $a \in A$, if $X_a = \{H \in \mathscr{P}_\kappa A | a \in H\}$, then $X_a \in \mu$. Now $jX_a = \{H \in \mathscr{P}_{\kappa'}(jA) | ja \in H\}$, where $\kappa' = j\kappa$. Obviously $ja \in j''A$, so to get $j``A \in jX_a$ it will suffice to see that $j``A \in \mathscr{P}_\kappa(jA)$. Evidently if $a \in A$ then $ja \in jA$, so $j``A \subseteq jA$. Since $\operatorname{card} j``A = \operatorname{card} A$ and $A \in R_\lambda \in R_{\kappa'}$, we must in fact have $j``A \in \mathscr{P}_\kappa(jA)$. Thus $\mu$ is a cover.

To see that $\mu$ has degree $\kappa$, note that if $H \in \mathscr{P}_\kappa A$, then $\bigcap\{X_a | a \in H\} \in \mu$. Thus if $\alpha < \kappa$, $\{H \mid \operatorname{card} H = \operatorname{card} \alpha\} \notin \mu$, and we have $\kappa$ sets of measure 0 whose union is the whole space.

It remains to see that the cover is irreducible. Suppose $X = \{x \in \mathscr{P}_\kappa A | gx \in x\} \in \mu$. Then $jX = \{x \in \mathscr{P}_\kappa(jA) | (jg)x \in x\}$, and since $j``A \in jX$, $(jg)(j``A) \in j``A$, i.e., there is $a \in A$ such that $(jg)(j``A) = ja$. But this means that $j``A \in jY$, where $Y = \{x | g(x) = a\}$. Thus we indeed have an irreducible cover.

Remark 7.4. For those familiar with the linguistic definition of strong compactness, we sketch the proof of the existence of a cover of $A$ from the $\kappa$-compactness of $L_{\kappa,\kappa}$. Let $A \in R_\lambda$, $\lambda$ a limit ordinal. Let $T$ be the complete $L_{\kappa,\kappa}$ theory of $(R_\lambda, \in)$. Add to the language constants $c$, $d$ and sentences saying "$\nu < c < \kappa$" for each $\nu < \kappa$, and "$a \in d \subseteq A$ & $\operatorname{card} d < \kappa$" for $a \in A$. If $\zeta$ is the new set of sentences, clearly each subset of $\zeta$ of cardinality $<\kappa$ is consistent. Since the sentences include well-foundedness and extensionality, we may take the model of $\zeta$ to be a transitive set $(M, \in)$. Putting $jx = x^M$, we have $j:(R_\lambda, \in) \simeq \prec(M, \in)$ with $\kappa$ as critical point. There is just enough in $M$ so that the argument of the preceding theorem can be carried through to show that $A$ has a cover (use $d^M$ instead of $j``A$ to induce the ultrafilter). Thus compactness is actually characterized by the existence of $j$, $M$ such that for some $d \in M$, $j``A \subseteq d \subseteq jA$ and $(\operatorname{card} d)^M < j\kappa$. (The argument from a cover to such an $M$ can be given directly using an ultrapower of $R_\lambda$ for $M$.) The only additional thing needed for an irreducible cover is that the type "$x \neq a$ & $x \in d$" $(a \in A)$ be omitted. Thus it seems that supercompactness adds some kind of type-omitting condition to compactness; how to formulate it precisely is not clear however. Supercompactness can also be characterized by the condition on $M$: One obtains such an $M$ from an ultrafilter $\mu$ on $\mathscr{P}_\kappa A$ by taking the ultrapower $R_\lambda^\mu$, which is isomorphic to such an $M$. The embedding is the canonical embedding into ultrapowers, and it turns out that the identity function on $\mathscr{P}_\kappa A$ represents $j``A$.

**8. Using fewer Ω-classes**. In order to express the idea of projecting imaginary sets and classes we made free use of Ω-classes. Now we try to be more efficient. First there is a rather trivial way we can use fewer Ω-classes: We can weaken our representation of the projected universe of imaginary objects, and use simply a well-founded transitive model $M$ instead of a natural model $R_{\lambda'}$. Let us look at the case $\lambda = \Omega + 1$. If $R_{\Omega+1} \cong \prec R_{\Omega'+1}$ then by the downward Löwenheim-Skolem theorem and the collapsing lemma there is an $M$ of cardinality $2^\Omega$ with $R_{\Omega+1} \subseteq M$ and $R_{\Omega+1} \cong \prec M \cong \prec R_{\Omega'+1}$, and $M$ is therefore a model of (S4) and the accompanying comprehension schema. Now we can suppose the elements of $M$ to be generated by a countable number of Skolem functions, and thus to be represented by terms with sets and classes as parameters, all of which can of course be construed as elements of $R_{\Omega+1}$. It thus seems that in a sense the ontology is limited to classes, while obtaining the benefits of thinking of proper classes as sets (with the ensuing higher type objects). The crucial thing however is the relation of membership among the terms, and a treatment of this requires a further type level. However in this way we can express a significant part of the idea behind (S4) using the universe $R_{\Omega+2}$. This is somewhat analogous to (4.2).

We easily see that there are ordinals $\kappa_0 < \kappa_1 < \Omega$ such that $E^1(\kappa_0 ; \kappa_1)$ and $E^1(\kappa_1 ; \Omega)$, which yields an analogue of (4.1): Choose $\Omega'$, $j$ so $j : R_{\Omega+2} \cong \prec R_{\Omega'+2}$, $\Omega$ critical point. In $R_{\Omega'+2}$ it is expressible, and true, that $E^1(\Omega ; \Omega')$. Thus $\exists \kappa E^1(\kappa ; \Omega')$ must be true in $R_{\Omega+2}$ also, i.e., $\exists \kappa_0 < \Omega E^1(\kappa_0 ; \Omega)$. Repeating the argument yields $\kappa_1$ with the desired properties. In fact, if $\mu$ is the normal measure on $\kappa$ given by

$$X \in \mu \leftrightarrow j``\kappa = \kappa \in jX,$$

then $\{\kappa \in \Omega | E^1(\kappa ; \Omega)\} \in \mu$. Similarly if $\mu$ on $\mathscr{P}_\kappa(R_{\Omega+1})$ is induced by $j``(R_{\Omega+1})$, then $\{H \subseteq R_{\Omega+1} | (\exists \kappa \in \Omega) R_{\kappa+1} \cong H \prec R_{\Omega'+1}\} \in \mu$. Similar results hold for higher types of extendibility. Note that $\kappa_0$ provides an interpretation of $\Omega$ which appears completely adequate for second-order statements.

**9. Appraisal.** Some comments about what we have tried to accomplish in the introduction of the concept of extendible cardinal (through § 6.2) may be helpful. We have tried to indicate that Cantor's $\Omega$ is extendible. It is difficult to describe the epistemological force of this indication; whether for example it should be called a plausibility argument or an informal proof, or just a proposal for a theory abouts sets and properties of sets. Ordinarily a plausibility argument can represent either the understanding of a proof, if you already have the proof, or a heuristic for finding a proof, if you do not yet have one (or a mistake if it is simply wrong). But in the case of an axiom, what does a plausibility argument represent? Perhaps some insight, or structured collection of insights, into the workings of some concepts (somewhat like imaginary experiments). In this case the term proof does not seem inappropriate if the insights are sharp enough. However, such informal proofs do not always carry conviction. Rather they are attempts to put in coherently organized form convictions we have perhaps together with some ideas about the way those convictions arise. While such convictions undoubtedly have an empirical content, we believe it is a gross exaggeration to say, for example, that we accept ZF because experience has not led to a contradiction (neither has NF). Our experience with ZF has perhaps as much to do with the ideas of ZF and the development of our intuition about sets as with the formal manipulations we make using the theory. There is also the element of appealing to the meanings of concepts, and the judgment of naturalness in elaborating and generalizing such meanings, and the recognition of analogies. The process by which we decide that some axioms (such as ZF) are correct, or have correct consequences, is of course not well understood. (It does not seem to me that of itself this casts any doubt on the validity of the decisions. Skepticism applied at the right point always has at least the virtue of preventing hubris, but applied at the wrong point it may not be beneficial.) Part of the process may involve getting experience with the axioms (such as the results Solovay has obtained concerning the GCH using large cardinals, or even the simple observations of §§ 7,8). Nevertheless, there must be a first step in recognizing axioms, and the foregoing introduction attempts to take such a first step, a step which will make the axioms seem worth considering as axioms rather than merely as conjectures or speculations. Thus we have attempted more than simply to motivate the mathematical concept of extendible cardinal. Since success in such informal ventures is not measured by the outcome of a known recursive procedure, the attempt may miss its mark. But the potential gain seems worth the risk; we must always remember that the incompleteness theorems show that large cardinal considerations have a bearing even on arithmetical problems.

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